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第二章 位势方程

$\Delta u = f$ in Ω $\begin{cases} f=0 & \text{Laplace} \\ f\neq 0 & \text{Poisson} \end{cases}$

u 仅为 x 函数

Dirichlet $u|_{\partial\Omega} = \varphi$

Neumann $\frac{\partial u}{\partial n}\big|_{\partial\Omega} = \varphi$

Robin $\frac{\partial u}{\partial n} + \sigma u\big|_{\partial\Omega} = \varphi$

Rmk. 波方程 $\partial_t^2 u - \Delta u = f$，$u$ 若不随 t 变化，则化为位势方程

位势方程可视作波动方程稳态解

§2.1 调和函数

def 1. 调和函数

$u: \Omega \to R$，有2阶连续偏导数，$\Delta u = 0$

def. u 为调和函数

Recall. 公式

1. $f: R^n \to R$

$$\int_{R^n} f(x)\,dx = \int_0^\infty \int_{\partial B(x_0,r)} f(y)\,dS(y)\,dr$$

$$(y = x_0 + rw) = \int_0^\infty \int_{|w|=1} f(x_0+rw)\,dS(w)\,r^{n-1}\,dr$$

$$\int_{B_{r_0}(x_0)} f(x)\,dx = \int_0^{r_0} \int_{|w|=1} f(x_0+rw)\,dS(w)\,r^{n-1}\,dr$$

2. $\frac{d}{dr}\int_{B(x_0,r)} f(y)\,dy = \int_{\partial B(x_0,r)} f(y)\,dS(y)$

def 2. 平均值性质

$u \in C(\Omega)$，u 满足平均值：

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$\forall B_r(x) \subset \Omega$, 有 $u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy$ ①

$\Omega \in \mathbb{R}^3$, $|B_r(x)| = \frac{4}{3}\pi r^3$

u满足第二平均值:

$\forall B_r(x) \subset \Omega$, 有 $u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y)$ ②

$\Omega \in \mathbb{R}^3$, $|\partial B_r(x)| = 4\pi r^2$, $u(x) = \frac{1}{4\pi}\int_{|w|=1} u(x+rw)\,dw$

claim: ① <=> ②

② => ①:

$$\forall B_r(x) \subseteq \Omega,\quad \int_{B_r(x)} u(y)\,dy = \int_0^r \left(\int_{\partial B(x,\rho)} u(y)\,dS(y)\right)d\rho$$
$$\overset{②}{=} \int_0^r |\partial B(x,\rho)|\, u(x)\,d\rho$$
$$= u(x)\int_0^r |\partial B(x,\rho)|\,d\rho = u(x)\,|B(x,r)| \Rightarrow ①$$

① => ②: (注意到: $|B(x,r)| = \int_0^r |\partial B(x,\rho)|\,d\rho$)

$\forall B_r(x) \subseteq \Omega$, $|B_r(x)|\,u(x) = \int_{B_r(x)} u(y)\,dy$

对 r 微分, 有 $|\partial B(r,x)|\,u(x) = \int_{\partial B(x,r)} u(y)\,dS(y)$ <=> ②

thm 1. 调和函数 => 平均值

$u \in C^2(\Omega)$, 为Ω上调和函数, 对任意 $B_r(x) \subseteq \Omega$,

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y)$$

pr: $\Delta u = 0$. in Ω $\forall B_r(x) \subset \Omega$

$$0 = \int_{B_r(x)} \Delta u\,dy = \int_{B_r(x)} \operatorname{div}(\nabla u)\,dy = \int_{\partial B(x,r)} \nabla u\cdot\vec{n}\,dS(y)$$
$$= \int_{|y-x|=r} \nabla u(y)\cdot\frac{y-x}{r}\,dS(y)$$
$$\overset{y=x+rw}{=} \int_{|w|=1} w\cdot\nabla u(x+rw)\,r^{n-1}\,dw \quad (w表示单位球面上面积元)$$
$$= r^{n-1}\int_{|w|=1} \frac{d}{dr}(u(x+rw))\,dw$$

$$= r^{n-1}\frac{d}{dr}\int_{|w|=1} u(x+rw)\,dw$$

$$\Rightarrow \forall r,\ \int_{|w|=1} u(x+rw)\,dw = \int_{|w|=1} u(x)\,dw = |\partial B(x,1)|\,u(x)$$

$$故\ u(x) = \frac{1}{|\partial B(x,1)|}\int_{|w|=1} u(x+rw)\,dw$$

$$\overset{y=x+rw}{=} \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y)$$

Thm 2. 平均值 ⇒ 调和、光滑

$u\in C(\Omega)$，满足 $\forall B_r(x)\subset\Omega$，

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y)$$

则 u 光滑且为调和函数

Recall 卷积

$$(f*g)(x) \overset{def.}{=} \int f(x-y)g(y)\,dy$$

f、g 中若一者光滑，则 $(f*g)(x)$ 光滑

pr：令 $\varphi\in C_0^\infty(B_1(0))$ (在一个紧集外恒为0)

$$\int_{\mathbb{R}^n}\varphi(x)\,dx = 1 = \int_{B_1(0)}\varphi(x)\,dx$$

$\varphi(x)=\varphi(|x|)$ (镜像、球面对称，仅为 r 函数，与角度无关)

$$则\ \int_0^1\int_{|w|=1}\varphi(r)\,r^{n-1}\,dw\,dr$$

$$= w_n\int_0^1\varphi(r)\,r^{n-1}\,dr$$ (w_n 表示 n 维单位球面面积)

$$=1$$ (利用了径向对称)

def：$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$

$\varphi_\varepsilon\in C_0^\infty(B_\varepsilon(0))$

$$且\ \int_{\mathbb{R}^n}\varphi_\varepsilon(x)\,dx = \int_{\mathbb{R}^n}\frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)dx \overset{x=\varepsilon y}{=} \int_{\mathbb{R}^n}\varphi(y)\,dy = 1$$

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claim:

$\forall x\in\Omega$, 取 $\varepsilon<\frac{1}{4}\mathrm{dist}(x,\partial\Omega)$, 则

$u(x)=(u*\varphi_\varepsilon)(x)$

$$(u*\varphi_\varepsilon)(x)=\int_\Omega u(y)\varphi_\varepsilon(x-y)\,dy$$
$$=\int_{\Omega\cap B_\varepsilon(x)}u(y)\frac{1}{\varepsilon^n}\varphi\left(\frac{x-y}{\varepsilon}\right)dy$$
$$=\int_{B_\varepsilon(x)}u(y)\frac{1}{\varepsilon^n}\varphi\left(\frac{x-y}{\varepsilon}\right)dy$$
$$\overset{y=x+rw}{=}\int_0^\varepsilon\int_{|w|=1}u(x+rw)\frac{1}{\varepsilon^n}\varphi\left(\frac{r}{\varepsilon}\right)r^{n-1}\,dr\,dw$$
$$\overset{\frac{r}{\varepsilon}=\rho}{=}\int_0^1\int_{|w|=1}u(x+\rho\varepsilon w)\,dw\,\rho^{n-1}\varphi(\rho)\,d\rho$$

$$\int_{|w|=1}u(x+\rho\varepsilon w)\,dw=\int_{|w|=1}u(x)\,dw=\omega_n u(x)$$

故 $(u*\varphi_\varepsilon)(x)=\omega_n\int_0^1\rho^{n-1}\varphi(\rho)\,d\rho\, u(x)=u(x)$

φ_ε 光滑, 由卷积性质 u 光滑

下说明 u 调和

step 1 $u\in C^2(\Omega)$

step 2 $\Delta u=0$

claim: $\forall x, r>0$, $\int_{B_r(x)}\Delta u(y)\,dy=0$

则 $\Delta u\equiv 0$

否则, $\exists x_0$. st. $(\Delta u)(x_0)\neq 0$, 不妨 $=c>0$

则 $\exists r_0>0$. $(\Delta u)(x)>\frac{c}{2}$ $\forall x\in B_{r_0}(x_0)$

则 $\int_{B_{r_0}(x_0)}\Delta u(y)\,dy\geq\frac{c}{2}|B_{r_0}(x_0)|>0$

事实上, $\int_{B_r(x)}\Delta u(y)\,dy=\int_{B_r(x)}\mathrm{div}(\nabla u)(y)\,dy=\int_{\partial B(x,r)}\nabla u\cdot\frac{y-x}{r}\,ds(y)$

$=\int_{|w|=1}w\cdot\nabla u(x+rw)r^{n-1}\,dw=r^{n-1}\frac{d}{dr}\int_{|w|=1}u(x+rw)\,dw\underset{\substack{\uparrow\\ \text{平均值性质}}}{=}0$

Rmk. $\int_{|w|=1}u(x+rw)\,dw$ 与 r 无关 $\Longleftrightarrow$ 平均值性质

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Rmk. 平均值性质即函数值 = 邻域内/球面上函数值的平均

满足平均值性质仅需可积

满足调和需要二阶连续可微

调和函数必然光滑

thm 3. (Harnack 不等式)

设 u 为 Ω 上非负连续函数，u 调和，对 Ω 上任意联通紧集 V，

$\exists\ C = C(\text{dist}(V, \partial\Omega), n)$, st. $\quad \text{dist}(V,\partial\Omega) = \inf\limits_{\substack{x\in V \\ y\in\partial\Omega}} |x-y|$

$$\sup_V u \le C \inf_V u$$

Rmk. C 与 V, n 有关，与函数 u 无关

pr: claim: $u(y) \le C u(x)$

令 $r = \frac{1}{4}\text{dist}(V, \partial\Omega)$

① $\forall x, y$ $|x-y| < r$，则 $B(x, 2r) \supset B(y, r)$

($\forall z \in B(y, r)$, $|y-z| < r$

$|z-x| \le |z-y| + |y-x| < 2r$, $z \in B(x, 2r)$)

u 调和，则 u 满足平均值性质

$$u(x) = \frac{1}{|B_{2r}(x)|}\int_{B_{2r}(x)} u(z)\,dz$$

$$\ge \frac{1}{2^n}\frac{1}{B_r(y)}\int_{B_r(y)} u(z)\,dz$$

$$= \frac{1}{2^n}u(y) \iff u(y) \le 2^n u(x)$$

② $\forall x, y \in V$

$\exists B_1, \cdots, B_N$, $B_i \cap B_{i+1} \neq \phi$，半径为 $\frac{r}{2}$，$x \in B_1$, $y \in B_N$

$\forall x_1, x_2 \in B_i$，有 $|x_1 - x_2| < r$，由①，$u(y) \le 2^{nN} u(x)$

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N 仅与 V 有关

由 x, y 任意性，$\sup\limits_{x\in V} u \leq 2^{nN} \inf\limits_{x\in V} u$

Rmk. 有限覆盖来源于 $\bigcup\limits_{x\in V} B_{\frac{r}{2}}(x)$

thm 4. (梯度估计)

$u \in C(\overline{B_R(x_0)})$ 是调和的，则

$|\nabla u(x_0)| \leq \frac{n}{R} \max\limits_{\overline{B_R(x_0)}} |u|$ (梯度为最大分量)

pr: 设 $u \in C^1(\overline{B_R(x_0)})$

(u 调和 $\Rightarrow$ $u \in C^\infty(B_R(x_0))$ 但未知边界处光滑性)

u 调和，则 u 在 B_R 上光滑 ($\overline{B_R(x_0)} = \overline{B_R}$)

故 $\partial_{x_i} u$ 也调和

由平均值性质 $\partial_{x_i} u(x_0) = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \partial_{x_i} u(y) dy$

$= \frac{1}{|B_R(x_0)|} \int_{\partial B_R(x_0)} u n_i \, dS(y)$

$\Rightarrow |\partial_{x_i} u(x_0)| \leq \frac{1}{|B_R(x_0)|} \int_{\partial B_R(x_0)} |u(y)| dS(y)$

$\leq \max\limits_{\overline{B_R}} |u| \cdot \frac{n}{R}$

thm 5.

若 $u \in C(\overline{B_R})$ 为非负调和函数，则

$|\nabla u(x_0)| \leq \frac{n}{R} u(x_0)$

pr: $|\nabla u(x_0)| \leq \frac{1}{|B_R(x_0)|} \int_{\partial B_R(x_0)} u(y) \, dS(y) \underset{\uparrow}{=} \frac{|\partial B(R,x_0)|}{|B_R(x_0)|} u(x_0) = \frac{n}{R} u(x_0)$

平均值性质

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Cor. (Liouville)

R^n上的上有界/下有界调和函数为常数

pr: $u \le M \Rightarrow u = const$

令 $v = M - u$，则 v 调和，$v \ge 0$

$\forall x_0, R>0$，对 v，$|\nabla v(x_0)| \le \frac{n}{R} v(x_0) = \frac{n}{R}(M - u(x_0))$

令 $R \to \infty$，$|\nabla v(x_0)| = 0$，则 $\nabla v \equiv 0$，即 v 为常数

§2.2 基本解和 Green 函数

def 1. 基本解

若 $\Delta u=\delta$，称 u 为基本解

δ 类似于电点荷，δ 为算子，$<\delta, f>=f(0)$

特别地，$\Delta u=0$，$\forall x\neq 0$

想先考虑特殊的解的形式

径向对称 $u(x)=u(|x|)$

$\Delta u=\partial_r^2 u+\frac{n-1}{r}\partial_r u+\frac{1}{r^2}\Delta_S u=\partial_r^2 u+\frac{n-1}{r}\partial_r u$

令 $v=\partial_r u$，则 $\partial_r v+\frac{n-1}{r}v=0$

$\Rightarrow v=C_1 r^{-(n-1)}$

$$\Rightarrow u(r)=\begin{cases} C_1 r^{-(n-2)}+C_2 & n\geq 3 \\ C_1 \ln r+C_2 & n=2 \end{cases}$$

def 2. 基本解

对 $x\in\mathbb{R}^n$，$x\neq 0$，$\Gamma(x)=\begin{cases} \frac{-1}{2\pi}\ln|x| & n=2 \\ \frac{1}{n(n-2)\alpha_n}\frac{1}{|x|^{n-2}} & n\geq 3 \end{cases}$

Rm. $\Delta\Gamma=\delta\neq 0$

thm 1. (Green)

$u\in C^2(\Omega)\cap C^1(\bar{\Omega})$，则

$$\int_\Omega u\Delta v\,dx=\int_\Omega \nabla\cdot(u\nabla v)-\nabla u\cdot\nabla v\,dx=\int_{\partial\Omega}u\nabla v\cdot\vec{n}\,ds-\int_\Omega\nabla u\cdot\nabla v\,dx$$
$$=\int_{\partial\Omega}u\cdot\frac{\partial v}{\partial n}ds-\int_\Omega\nabla u\cdot\nabla v\,dx \quad \text{(第一 Green)}$$

$$\int_\Omega u\Delta v-v\Delta u\,dx=\int_{\partial\Omega}u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\,ds \quad \text{(第二 Green)}$$

进一步考虑方程解的表达形式

thm 2. 三维下 $\Delta u=0$ in Ω 的解

$u\in C^2(\Omega)\cap C(\bar\Omega)$, $\Delta u=0$ in Ω

$\forall x_0\in\Omega$, 有 $u(x_0)=\frac{1}{4\pi}\int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\frac{\partial u}{\partial n}\right]ds(x)$ (※)

pr: 尝试利用 Green 公式，但 $\frac{1}{|x-x_0|}$ 具有奇性

在 $\Omega_\varepsilon=\Omega\setminus\overline{B_\varepsilon(x_0)}$ 用第二 Green 公式

事实上 Laplace 方程满足平移不变性，则不妨设 $x_0=0$

只需证：若 $0\in\Omega$, $u(0)=\frac{1}{4\pi}\int_{\partial\Omega}\left[-u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}\right]ds(x)$ (※※)

若 (※)(※) 成立，对 $u(\cdot+x_0)$ 应用 (※※)

$$u(x_0)=\frac{1}{4\pi}\int_{\partial(\Omega-x_0)}\left[-u(x+x_0)\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{|x|}\frac{\partial u}{\partial n}(x+x_0)\right]ds$$
$$=\frac{1}{4\pi}\int_{\partial\Omega}\left[-u(x)\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{|x-x_0|}\frac{\partial u}{\partial n}\right]ds$$

在 $\Omega_2=\Omega\setminus\overline{B_\varepsilon(0)}$ 上应用第二 Green，对 u. $v=\frac{1}{4\pi|x|}$

$$\int_{\Omega_2}(u\Delta v-v\Delta u)dx=0$$
$$=\int_{\partial\Omega_2}u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\,ds$$
$$=\frac{1}{4\pi}\int_{\partial\Omega}u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)-\frac{1}{|x|}\frac{\partial u}{\partial n}\,ds-\underbrace{\frac{1}{4\pi}\int_{\partial B_\varepsilon}u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)-\frac{1}{|x|}\frac{\partial u}{\partial n}\,ds}_{\triangleq I}$$

$$I=\frac{1}{4\pi}\int_{|x|=\varepsilon}u\cdot\frac{-1}{r^2}\,ds-\frac{1}{4\pi}\int_{|x|=\varepsilon}\frac{1}{|x|}\frac{\partial u}{\partial n}\,ds$$
$$=\frac{-1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(x)\,ds-\frac{1}{4\pi\varepsilon}\int_{B(r,\varepsilon)}\Delta u\,dx$$

令 $\varepsilon\to0$, $I=-u(0)$

Rmk. 也可利用估计

$$\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(x)-u(0)\,ds\le\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}|(\nabla u)(\xi)||x|\,ds\le\max_{\xi\in\bar\Omega}|(\nabla u)(\xi)|\underbrace{\frac{\varepsilon}{4\pi\varepsilon^2}\cdot4\pi\varepsilon^2}_{\to0}$$

$$\frac{1}{4\pi}\int_{|x|=\varepsilon}\frac{1}{|x|}\frac{\partial u}{\partial n}ds\le\max_{x\in\Omega}\left|\frac{\partial u}{\partial n}\right|\frac{1}{4\pi\varepsilon}\cdot4\pi\varepsilon^2\to0$$

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需要 u，$\frac{\partial u}{\partial n}$ 在边界的值，但通常不给定

若 g 在 Ω 内调和，$g|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}$，则对 u，g 在 Ω 上用第二 Green

$$0 = \int_{\partial\Omega} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) ds$$

$$\Rightarrow u(x_0) = \int_{\partial\Omega} \left[u\frac{\partial}{\partial n}\left(g - \frac{1}{4\pi|x-x_0|}\right) - \left(g - \frac{1}{4\pi|x-x_0|}\right)\frac{\partial u}{\partial n}\right] ds$$

$$= \int_{\partial\Omega} u\frac{\partial}{\partial n}\left(g - \frac{1}{4\pi|x-x_0|}\right) ds$$

令 $G(x,x_0) = -g(x) + \frac{1}{4\pi|x-x_0|}$

则 $u(x_0) = -\int_{\partial\Omega} u(x)\frac{\partial G}{\partial n}(x,x_0)\, dS(x)$　　Poisson 公式 (2.1)

g 的表达形式应同样被给出

def 3. Green 函数

Ω 上的算子 $-\Delta$ 的 Green 函数，满足

(1) $G(x)$ 在 Ω 内除 x_0 点外二阶连续可微且调和

(2) $G(x) = 0$，$\forall x \in \partial\Omega$

(3) $-G(x) + \frac{1}{4\pi|x-x_0|}$ 在 x_0 有限，且处二阶连续可微且调和

thm3. 性质：$G(x,x_0) = G(x_0,x)$

对 $u = G(x,a)$，$v = G(b,x)$ 在 $\Omega_\varepsilon = \Omega\setminus(\overline{B_\varepsilon(a)} \cup \overline{B_\varepsilon(b)})$ 应用第二 Green

则 $0 = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$

$$= \underbrace{\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds}_{\overset{(2)}{=}\,0} + \underbrace{\int_{|x-a|=\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds}_{\triangleq\, A_\varepsilon} + \underbrace{\int_{|x-b|=\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds}_{\triangleq\, B_\varepsilon}$$

$$A_\varepsilon = \underbrace{\int_{|x-a|=\varepsilon}\left[\left(u - \frac{1}{4\pi|x-a|}\right)\frac{\partial v}{\partial n} - v\frac{\partial}{\partial n}\left(u - \frac{1}{4\pi|x-a|}\right)\right]ds}_{①} + \underbrace{\int_{|x-a|=\varepsilon}\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}\,ds}_{②} - \underbrace{\int_{|x-a|=\varepsilon} v\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-a|}\right)ds}_{③}$$

$$① = -\int_{|x-a|<\varepsilon}\left[\Delta\left(u - \frac{1}{4\pi|x-a|}\right)v - \left(u - \frac{1}{4\pi|x-a|}\right)\Delta v\right]dx = 0$$

$$② = \frac{1}{4\pi\varepsilon}\int_{|x-a|=\varepsilon}\frac{\partial v}{\partial n}\,ds = \frac{1}{4\pi\varepsilon}\int_{|x-a|<\varepsilon}\Delta v\,dx = 0$$

$$③ = \int_{|x-a|=\varepsilon} v\frac{\partial}{\partial r}\left(\frac{1}{4\pi r}\right)ds = \frac{-1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon} v\,ds \to -v(a) = -G(b,a)$$

$$\left.\begin{array}{l}\text{即 } A_\varepsilon \to -G(b,a) \\ \text{同理 } B_\varepsilon \to G(a,b)\end{array}\right\} \Rightarrow G(a,b) = G(b,a)$$

若给出了G的表达，则给出了g的表达，下面考查Green函数求法

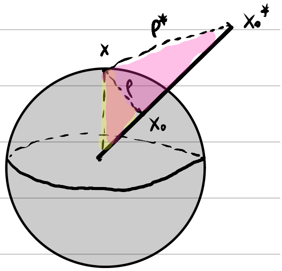
1. 半空间

取 $x_0^* = (x_0^1, x_0^2, -x_0^3)$

def $G(x,x_0) = \frac{1}{4\pi|x-x_0|} - \frac{1}{4\pi|x-x_0^*|}$

则 G 满足 (1)(2)(3)

2. $B_R(0)$



$G(x,x_0) = \frac{1}{4\pi|x-x_0|} - \frac{C}{4\pi|x-x_0^*|}$

x_0^* 在球外，此时 (1)(3) 自然满足

由(2)，$\forall |x| = R$，有 $G(x,x_0) = 0$

令 $|x-x_0| = \rho$，$|x-x_0^*| = \rho^* \Rightarrow \frac{\rho^*}{\rho} = C$

若 Δ小 ∼ Δ大，则 $\frac{\rho^*}{\rho} = \frac{|x_0^*|}{R} = \frac{R}{|x_0|}$

取 x_0^* 满足 $x_0^* = \frac{R^2}{|x_0|^2}x_0$ 即可

则 $G(x,x_0) = \frac{1}{4\pi|x-x_0|} - \frac{C}{4\pi|x-x_0^*|}$

$C = \frac{R}{|x_0|}$，$x_0^* = \frac{R^2}{|x_0|^2}x_0$ 即可

将 $B_R(0)$ 下的Green函数带入(2.1)，$x\in\partial\Omega$ 时

$$\nabla G(x,x_0) = -\frac{x-x_0}{4\pi|x-x_0|^3} + \frac{C(x-x_0^*)}{4\pi|x-x_0^*|^3} = -\frac{x-x_0}{4\pi|x-x_0|^3} + \frac{|x_0|^2}{4\pi R^2}\frac{x-x_0^*}{|x-x_0|^3}$$

$$= -\frac{x}{4\pi|x-x_0|^3}\left(1-\frac{|x_0|^2}{R^2}\right) + \frac{1}{4\pi|x-x_0|^3}\left(x_0 - \frac{|x_0|^2}{R^2}x_0^*\right) = -\frac{x}{4\pi|x-x_0|^3}\left(1-\frac{|x_0|^2}{R^2}\right)$$

$$\frac{\partial G}{\partial n}=n\cdot\nabla G=\frac{x}{|x|}\cdot\left(\frac{R^2-|x_0|^2}{R^2}\frac{x}{4\pi|x-x_0|^3}\right)=\frac{R^2-|x_0|^2}{R}\frac{1}{4\pi|x-x_0|^3}$$

故 $u(x_0)=\frac{R^2-|x_0|^2}{4\pi R}\int_{|x|=R}\frac{\varphi(x)}{|x-x_0|^3}dS(x)$ （φ为边值）

即 $u(x)=\frac{R^2-x^2}{4\pi R}\int_{|y|=R}\frac{\varphi(y)}{|x-y|^3}dS(y)$　　Poisson公式

thm 4.（Harnack不等式）

设 u 在 $B_R(x_0)$ 内调和且 $u\geq 0$，则

$$\frac{R}{R+r}\frac{R-r}{R+r}u(x_0)\leq u(x)\leq\frac{R}{R-r}\frac{R+r}{R-r}u(x_0)，其中\ r=|x-x_0|<R$$

Rmk. 2.1中也有Harnack不等式，均说明连通紧集中任意两点处函数值可互相比较，区别在于该thm给出了具体的界

pr:　不妨设 $x_0=0$（平移不变性）

则 $r=|x|<R$，只需证：

$$\frac{R}{R+r}\frac{R-r}{R+r}u(0)\leq u(x)\leq\frac{R}{R-r}\frac{R+r}{R-r}u(0)$$

由Poisson公式，$u(x)=\frac{R^2-x^2}{4\pi R}\int_{|y|=R}\frac{u(y)}{|x-y|^3}dS(y)$

由于 $R-r\leq|x-y|\leq R+r$，则

$$\frac{R^2-r^2}{4\pi R}\frac{1}{(R+r)^3}\int_{|y|=R}u(y)dS(y)\leq u(x)\leq\frac{R^2-r^2}{4\pi R}\frac{1}{(R-r)^3}\int_{|y|=R}u(y)dS(y)$$

u 调和，由平均值性质，则 $\int_{|y|=R}u(y)dS(y)=4\pi R^2u(0)$

$$\frac{R(R^2-r^2)}{(R+r)^3}u(0)\leq u(x)\leq\frac{R(R^2-r^2)}{(R-r)^3}u(0)$$　即证

n维情形下的推导:

$$\begin{cases} -\Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$$

对u、基本解为v，在 $\Omega \setminus B_\varepsilon(0) \triangleq \Omega_\varepsilon$ 上用第二Green公式

$$\int_{\Omega_\varepsilon} u\Delta v - v\Delta u \, dx = 0$$

$$= \int_{\partial\Omega_\varepsilon} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \, dS(x) = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \, dS(x) - \int_{\partial B(0,\varepsilon)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \, dS(x)$$

① $\int_{\partial B(0,\varepsilon)} u\frac{\partial v}{\partial n} \, dS(x)$

$$= \int_{\partial B(0,\varepsilon)} u\frac{\partial}{\partial n}\left(\frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}\right) dS(x)$$

$$= \int_{\partial B(0,\varepsilon)} u\frac{\partial}{\partial r}\left(\frac{1}{n(n-2)\alpha(n)} \frac{1}{r^{n-2}}\right) dS(x)$$

$$= \int_{\partial B(0,\varepsilon)} u \cdot \frac{1}{n(n-2)\alpha(n)} \frac{2-n}{r^{n-1}} \, dS(x)$$

$$= \frac{-1}{n\alpha(n)} \frac{1}{\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} u \, dS(x) \to -u(0) \ (\varepsilon \to 0)$$

② $\int_{\partial B(0,\varepsilon)} v\frac{\partial u}{\partial n} \, dS(x)$

$$= \int_{\partial B(0,\varepsilon)} \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} \frac{\partial u}{\partial n} \, dS(x)$$

$$= \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\varepsilon|^{n-2}} \int_{B(0,\varepsilon)} \Delta u \, dx = 0$$

故 $u(0) = \int_{\partial\Omega} -u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n} \, dS(x)$

若 $-\Delta u = f$，则 $\int_{\partial B(0,\varepsilon)} v\frac{\partial u}{\partial n} \, dS(x) = \frac{1}{n(n-2)\alpha(n)} \frac{-1}{|\varepsilon|^{n-2}} \int_{B(0,\varepsilon)} f \, dx$

有 $\int_{\Omega_\varepsilon} vf \, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \, dS(x) + u(0) + \frac{1}{n(n-2)\alpha(n)} \frac{-1}{|\varepsilon|^{n-2}} \int_{B(0,\varepsilon)} f \, dx$

$$\Rightarrow u(0) = \int_{\partial\Omega} -u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n} dS(x) + \int_{\Omega_\varepsilon} vf \, dx + \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\varepsilon|^{n-2}} \int_{B(0,\varepsilon)} f \, dx$$

$$= \int_{\partial\Omega} -u\frac{\partial v}{\partial n} + v\frac{\partial u}{\partial n} dS(x) + \int_{\Omega} vf \, dx$$

$$\Leftrightarrow u(x) = \int_\Omega f(y)\Gamma(x-y) \, dy - \int_{\partial\Omega} g(y)\frac{\partial\Gamma}{\partial n}(x-y) + \int_{\partial\Omega} \frac{\partial u}{\partial n}(y)\Gamma(x-y) \, dS(y)$$

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用 $G(x,y) \triangleq \Gamma(x-y) - \phi^x(y)$ 代替 $\Gamma(x-y)$

要求 $G\big|_{\partial\Omega} = 0$，$\Delta G(x,y) = 0$

$\Downarrow$　　　　　　$\Downarrow$

$\phi^x(y)\big|_{\partial\Omega} = \Gamma(x-y)$　　$\Delta\phi^x(y) = 0$

$$u(x) = \int_\Omega f(y) G(x-y)\, dy - \int_{\partial\Omega} \frac{\partial G}{\partial n}(x,y) g(y)\, dS(y)$$

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§2.3 极值原理和最大模估计

考虑方程 $\mathcal{L}u=-\Delta u+C(x)u=f(x)$, $C(x)\geq 0$, $x\in\Omega$

thm 1. 弱极大值原理

$C(x)\geq 0$, $f(x)<0$. $u\in C^2(\Omega)\cap C(\bar\Omega)$, 且满足以上方程, 则

$u(x)$ 不能在 Ω 上达到它在 $\bar\Omega$ 上的非负最大值,

即 $u(x)$ 只能在 $\partial\Omega$ 上达到非负最大值

(注意, 并不意味在边界上一定能取到非负最大值)

pr: 假设 u 在 $x_0\in\Omega$ 达到非负最大值 $M\geq 0$

则 $(\nabla u)(x_0)=0$, $(\Delta u)(x_0)=tr(\text{Hesse}\,u(x_0))\leq 0$

$(\mathcal{L}u)(x_0)=-\Delta u(x_0)+C(x_0)u(x_0)\geq 0$ 与 $f(x)<0$ 矛盾

thm 2.

$C(x)\geq 0$, $f(x)\leq 0$. $u\in C^2(\Omega)\cap C(\bar\Omega)$, 且满足以上方程

且在 $\bar\Omega$ 上存在正的最大值, 则 $u(x)$ 必在 $\partial\Omega$ 上到达在 $\bar\Omega$ 上的非负最大值

且 $\max\limits_{x\in\bar\Omega} u(x)\leq\max\limits_{\partial\Omega} u^+(x)$

$u^+=\max\{u(x),0\}$

pr: 不妨设 $0\in\Omega$, 令 $d=\text{diam}\,\Omega$.

令 $v(x)=-(d^2-|x|^2)\leq 0$ 　 $u^\varepsilon(x)=u(x)+\varepsilon v(x)$

$\mathcal{L}u^\varepsilon=\mathcal{L}u+\varepsilon\mathcal{L}v=f+\varepsilon(-\Delta(-d^2+|x|^2)+c(x)(-d^2+|x|^2))$

$=f-2n\varepsilon+\varepsilon c(x)(-d^2+|x|^2)<0$

应用 thm 1.

$\max\limits_{\bar\Omega}u-\varepsilon d^2=\max\limits_{\bar\Omega}(u-\varepsilon d^2)\leq\max\limits_{\bar\Omega}u_\varepsilon\leq\max\limits_{\partial\Omega}(u+\varepsilon v)^+\leq\max\limits_{\partial\Omega}u^+$

令 $\varepsilon\to 0$. $\max\limits_{\bar\Omega}u\leq\max\limits_{\partial\Omega}u^+$

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Rmk. 若u在Ω上最大值为负数，与thm 1.2无关

下面我们应证明强极大值原理.

thm 3. (Hopf引理)

设B_R为R^n ($n=2,3$)上以R为半径的球，在B_R上$c(x)\ge 0$有界，

若$u\in C^2(B_R)\cap C^1(\overline{B_R})$满足

(1) $\mathcal{L}u=-\Delta u+c(x)u\le 0$，$x\in B_R$

(2) $\exists x_0\in\partial B_R$，s.t. u在$x_0$处达到$\overline{B_R}$上的非负最大值

即 $u(x_0)=\max\limits_{\overline{B_R}} u\ge 0$　且当$x\in B_R$时，$u(x)<u(x_0)$，

则

$$\left.\frac{\partial u}{\partial \mu}\right|_{x=x_0}>0，$$ μ与∂B_R在x_0点单位外法向量n夹角小于$\frac{\pi}{2}$

pr: 由(2)易知 $\left.\frac{\partial u}{\partial \nu}\right|_{x=x_0}\ge 0$

令 $v(x)=e^{-\alpha|x|^2}-e^{-\alpha R^2}$　$\alpha>0$待定

$w(x)=u(x)-u(x_0)+\varepsilon v(x)$　$\varepsilon>0$

$\nabla v=e^{-\alpha|x|^2}\cdot(-2\alpha\vec{x})$

$$\Delta v=\sum_i \partial x_i(-2\alpha x_i e^{-\alpha|x|^2})=\sum_i(-2\alpha+4\alpha^2 x_i^2)e^{-\alpha|x|^2}$$
$$=(-2\alpha n+4\alpha^2|x|^2)e^{-\alpha|x|^2}$$

$$\mathcal{L}w=\mathcal{L}u-c(x)u(x_0)+\varepsilon\mathcal{L}v$$
$$=\mathcal{L}u-c(x)u(x_0)+\varepsilon\left[(-4|\alpha|^2|x|^2+2\alpha n)e^{-\alpha|x|^2}+c(x)(e^{-\alpha|x|^2}-e^{-\alpha R^2})\right]$$
$$=\underset{\le 0}{\mathcal{L}u}-\underset{\le 0}{c(x)u(x_0)}+\underset{>0}{\varepsilon}\left[(-4|\alpha|^2|x|^2+2\alpha n+c(x))e^{-\alpha|x|^2}-\underset{\ge 0}{c(x)e^{-\alpha R^2}}\right]$$
$$\le\varepsilon(-4|\alpha|^2|x|^2+2\alpha n+C)e^{-\alpha|x|^2}$$　C为c(x)的界

在 $B_R^* = \{\frac{R}{2} \le |x| \le R\}$ 上做估计,

$\le \varepsilon(-R^2\alpha^2 + 2\alpha n + c)e^{-\alpha|x|^2}$

< 0 (α充分大)

由 thm 1, 在 B_R^* 对 w 应用极大值原理,

w在 $\overline{B_R}$ 上的非负最大值必在边界上取到

$|x| = \frac{R}{2}$ 时, $w(x) = u(x) - u(x_0) + \varepsilon(e^{-\alpha\cdot\frac{R^2}{4}} - e^{-\alpha R^2})$

$$\Rightarrow w(x) \le \max_{|x|=\frac{R}{2}} u(x) - u(x_0) + \varepsilon(e^{-\alpha\cdot\frac{R^2}{4}} - e^{-\alpha R^2})$$

< 0 (ε充分小)

$|x| = R$时, $w(x)$ 在 x_0处取得最大值

$$\Rightarrow \frac{\partial w}{\partial \mu} \ge 0$$

即 $\frac{\partial u}{\partial \mu} + \varepsilon\frac{\partial v}{\partial \mu} \ge 0$

而 $\frac{\partial v}{\partial \mu} = \mu\cdot\nabla v = \mu\cdot e^{-\alpha|x|^2}(-2\alpha\bar{x}) < 0$

$$\Rightarrow \frac{\partial u}{\partial \mu} > 0$$

thm 4. 强极大值原理

假设 Ω 为 R^n 中有界连通开集, $c(x) \ge 0$ 有界

若 $u \in C^2(\Omega) \cap C(\bar{\Omega})$ 在 Ω 上满足 $Lu \le 0$, 且 u在 Ω内达到在 $\bar{\Omega}$上非负最大值, 则 u 在 $\bar{\Omega}$ 上恒为常数.

pr: 令 $M = \max_{x\in\bar{\Omega}} u(x) \ge 0$, 令 $O = \{x \in \Omega \mid u(x) = M\}$, 即证 $\Omega = O$

step 1. $O \ne \phi$.

step 2. O闭, 即 $x_n \in O$, $x_n \to \bar{x}$, 则 $\bar{x} \in O$

u连续, 则 $u(\bar{x}) = \lim_{n\to\infty} u(x_n) = M$, 则 $\bar{x} \in O$

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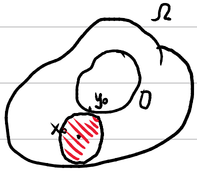
Step 3. O开.

若不为开集. 则$\exists\, x_0\in\Omega\setminus O$ (Ω开而O不开, $O\subsetneq\Omega$)

$\Omega\setminus O$开. $\exists\, R>0$. s.t. $B_R(x_0)$与$\partial\Omega$相切于y_0

claim: ① u在y_0达到$\bar\Omega$上的非严格最大值

② $\forall x\in B_R(x_0)$. $u(x)<M$



由Hopf引理, $\frac{\partial u}{\partial n}(y_0)>0$

($y_0\in\Omega$, 由于$\Omega^c\cap O=\phi$, Ω^c, O均闭. $dist(\Omega^c, O)>0$)

而$y_0\in\Omega$达到最大值, $\nabla u(y_0)=0$, $\frac{\partial u}{\partial n}(y_0)=0$矛盾

则O为开集

综合1.2.3知$O=\Omega$

Rm. ① 若$-\Delta u<0$. 则u只在边界处达到最大值.

(若在$x_0\in\Omega$处达到, 则$-\Delta u(x_0)\geq 0$矛盾)

② 若u调和 ($u\in C(\bar\Omega)$)(满足平均值性质). 则u只在$\partial\Omega$达到最大值和最小值, 除非u为常数(即若最大值/最小值在内部取到, 则必为常数)

pr: 不妨最大值在内部取到, 设$M=\max\limits_{\bar\Omega} u(x)$

令$O=\{x\in\Omega \mid u(x)=M\}$. O非空闭

只需证O为开集

$\forall x_0\in O$, $\exists\, B_R(x_0)\in\Omega$, 由平均值性质

$$M=u(x_0)=\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}u(y)\,dy\leq M$$

$\Rightarrow u(x)\equiv M$, $x\in B_R(x_0)$, 则$B_R(x_0)\subseteq M$

故O为开集

①②表明$\Delta u\geq 0$. 则仅能在边界处达到最大值, 无非负性要求.

下利用极值原理证明最大模估计.

考虑 $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$ (2.41)

thm 5. 最大模估计

$u \in C^2(\Omega) \cap C(\bar{\Omega})$ 为 (2.41) 的解，则

$$\max_{\bar{\Omega}} |u(x)| \le G + CF, \quad G = \max_{\partial\Omega} |g|, \quad F = \max_{\bar{\Omega}} |f|, \quad C = C(n, \mathrm{diam}(\Omega))$$

idea. $\Delta u \geq 0$, $u|_{\partial\Omega} \le 0 \Rightarrow u \le 0$

pr: 不妨 $0 \in \Omega$, $\forall x \in \Omega$, $|x| < d$

令 $w(x) = u - \left(G + \frac{F}{2n}(d^2 - |x|^2)\right)$

则 $-\Delta w = -\Delta u - F = f - F \le 0$

$w|_{\partial\Omega} = g - G - \frac{F}{2n}(d^2 - |x|^2) \le g - G \le 0$

由弱极值原理, $\max_{\bar{\Omega}} w - \max_{\partial\Omega} w \le 0$

$$0 \geq \max_{\bar{\Omega}} \left(u(x) - \left(G + \frac{F}{2n} d^2\right) + \frac{F}{2n}|x|^2 \right) \geq \max_{\bar{\Omega}} u(x) - \left(G + \frac{F}{2n} d^2\right)$$

$\Rightarrow u(x) \le G + \frac{d^2}{2n} F$

再令 $\tilde{w}(x) = -u - \left(G + \frac{F}{2n}(d^2 - |x|^2)\right)$

则 $-\Delta \tilde{w} = -f - F \le 0$

$\tilde{w}|_{\partial\Omega} \le -g - G \le 0$

由弱极值原理, $\max_{\bar{\Omega}} \tilde{w} - \max_{\partial\Omega} \tilde{w} \le 0$

$\Rightarrow -u(x) \le G + \frac{d^2}{2n} F$

故 $|u(x)| \le G + \frac{d^2}{2n} F \quad \forall x \in \Omega$.

Rmk. 最大模估计蕴含解的唯一性与稳定性.

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设 u_1、u_2 为 (2.41) 的解，设 $v = u_1 - u_2$

则 $\begin{cases} -\Delta v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$

由最大模估计 $\max\limits_{\bar\Omega} |v(x)| \le 0 + C\cdot 0 = 0 \Rightarrow u_1 = u_2$，即唯一性

设 u_1、u_2 为

$$\begin{cases} -\Delta u = f_1 \\ u|_{\partial\Omega} = g_1 \end{cases} \quad 与 \quad \begin{cases} -\Delta u = f_2 \\ u|_{\partial\Omega} = g_2 \end{cases}$$

的解

由最大模估计 $\max\limits_{\bar\Omega} |u_1(x) - u_2(x)| \le \max\limits_{\partial\Omega} |g_1 - g_2| + C \max\limits_{\bar\Omega} |f_1 - f_2|$

蕴含稳定性

而唯一性与稳定性可通过能量法给出

考虑方程 $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$$\int_\Omega u(-\Delta u)\,dx = \int_\Omega fu\,dx = \int_\Omega -\mathrm{div}(u\nabla u) + |\nabla u|^2\,dx = -\int_{\partial\Omega} u\frac{\partial u}{\partial n}\,ds + \int_\Omega |\nabla u|^2\,dx$$

claim. (Friednchs 不等式)

$u \in C_0^1(\Omega)$，则 $\int_\Omega |u(x)|^2 dx \le 4d^2 \int_\Omega |\nabla u(x)|^2 dx \quad d = \mathrm{diam}(\Omega)$

则 $\int_\Omega fu\,dx \le \varepsilon \int_\Omega |u|^2 dx + \frac{1}{4\varepsilon}\int_\Omega |f|^2 dx$

$$\int_\Omega |\nabla u|^2 dx \le \varepsilon \cdot 4d^2 \int_\Omega |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_\Omega |f|^2 dx$$

选取合适的 ε 可 s.t. $\int_\Omega |\nabla u|^2 dx \le C\cdot \int_\Omega |f|^2 dx$

Rmk. 该表达式蕴含 $|\nabla u|$、$|u|$ 的控制，Poisson 方程的能量为方程两边 $\times u$.

第三章 热传导方程

$u_t - \Delta u = f(x,t) \quad x\in\Omega,\ t>0$

$u(x,0) = \varphi(x)$

(Dirichlet) $u\big|_{\partial\Omega} = g(x,t)$

(Newmann) $\frac{\partial u}{\partial n}\big|_{\partial\Omega} = g(x,t)$

(Robin) $\frac{\partial u}{\partial n} + \sigma u\big|_{\partial\Omega} = g(x,t)$

u表示温度 etc，描述传热过程、扩散过程；

f表示热源

§3.1 初值问题

解法1 分离变量法

$\Omega = [0,l]$、矩形区域、圆盘 etc.

解法2. Fourier变换法

考察方程 $\begin{cases} u_t - \Delta u = f(x,t) & x\in\mathbb{R}^n,\ t>0 \\ u(x,0) = \varphi(x) \end{cases}$

Recall、($\mathbb{R}^n$上的Fourier变换)

$f\in L^1(\mathbb{R}^n)$, $\left(\int_{\mathbb{R}^n}|f(x)|\,dx < +\infty\right)$

def. $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx$

性质 $f\in \mathcal{S}(\mathbb{R}^n)$, ($f$光滑, $\forall$阶导数衰减得很快)

则不考虑分部积分边界项

1、令 $(\tau_{x_0}f)(x) = f(x-x_0)$，则 $\widehat{\tau_{x_0}f}(\xi) = e^{-2\pi i x_0\cdot\xi}\hat{f}(\xi)$

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pr. $\widehat{\tau_{x_0} f}(\xi) = \int_{\mathbb{R}^n} f(x-x_0) e^{-2\pi i x\cdot\xi} dx$

$= \int_{\mathbb{R}^n} f(y) e^{-2\pi i (x_0+y)\cdot\xi} dy$

$= e^{-2\pi i x_0\cdot\xi} \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi} dy$

$= e^{-2\pi i x_0\cdot\xi} \hat{f}(\xi)$

2. 令 $(S_\lambda f)(x) = f(\lambda x)$，则 $\widehat{S_\lambda f}(\xi) = \lambda^{-n} \hat{f}(\lambda^{-1}\xi)$

pr. $\widehat{S_\lambda f}(\xi) = \int_{\mathbb{R}^n} f(\lambda x) e^{-2\pi i x\cdot\xi} dx$

$= \int_{\mathbb{R}^n} f(y) e^{-2\pi i \frac{y}{\lambda}\cdot\xi} \lambda^{-n} dy$

$= \lambda^{-n} \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\frac{\xi}{\lambda}} dy$

$= \lambda^{-n} \int_{\mathbb{R}^n} \hat{f}(\lambda^{-1}\xi)$

3. 对多重指标 $\alpha = (\alpha_1, \cdots, \alpha_n)$，记 $|\alpha| = \alpha_1 + \cdots + \alpha_n$，$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\partial^\alpha = \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$，则 $\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$

pr. $\widehat{\partial x_j f}(\xi) = \int_{\mathbb{R}^n} f_{x_j}(x) e^{-2\pi i x\cdot\xi} dx$

$= -\int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} (-2\pi i \xi_j) dx$

$= 2\pi i \xi_j \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx$

$= 2\pi i \xi_j \hat{f}(\xi)$

4. $\widehat{(-2\pi i x)^\alpha f}(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$

pr. $\widehat{-2\pi i x_j f}(\xi) = \int_{\mathbb{R}^n} -2\pi i x_j f(x) e^{-2\pi i x\cdot\xi} dx$

$= \int_{\mathbb{R}^n} (e^{-2\pi i x\cdot\xi})_{\xi_j} f(x) dx$

$= \partial_{\xi_j} \int_{\mathbb{R}^n} f(x) e^{-2\pi i \cdot\xi} dx$

$= \partial_{\xi_j} \hat{f}(\xi)$

5. 令 $(f*g)(x)=\int_{\mathbb{R}^n} f(x-y)g(y)\,dy=\int_{\mathbb{R}^n} f(y)g(x-y)\,dy$

$\forall f,g\in \mathcal{S}(\mathbb{R}^n)$，则 $\widehat{f*g}(\xi)=\hat{f}(\xi)\hat{g}(\xi)$

6、逆变换

$\check{f}(x)=\int_{\mathbb{R}^n} f(\xi)e^{2\pi i x\cdot\xi}d\xi$，则若 $f\in\mathcal{S}(\mathbb{R}^n)$，有

$\hat{f}\in\mathcal{S}(\mathbb{R}^n)$，$\check{\hat{f}}=f$

例. 若 $f(x)=e^{-x^2}$，$x\in\mathbb{R}$，则 $\hat{f}(\xi)=\sqrt{\pi}\,e^{-\pi^2\xi^2}$

令 $F(\xi)=\hat{f}(\xi)=\int_{\mathbb{R}} e^{-x^2}e^{-2\pi i x\xi}dx$

$F'(\xi)=\int_{\mathbb{R}} e^{-x^2}(-2\pi i x)e^{-2\pi i x\xi}dx$

$=\pi i\int_{\mathbb{R}} (e^{-x^2})' e^{-2\pi i x\xi}dx$

$=\pi i\hat{f'}(\xi)=\pi i\cdot 2\pi i\xi\hat{f}(\xi)$

$=-2\pi^2\xi F(\xi)$

$F(0)=\int_{\mathbb{R}} e^{-x^2}dx=\sqrt{\pi}$

$\Rightarrow F(\xi)=\sqrt{\pi}\,e^{\int -2\pi^2\xi d\xi}=\sqrt{\pi}\,e^{-\pi^2\xi^2}$

注：$f(x)\in\mathcal{S}(\mathbb{R})$，则 $(e^{-\pi^2\xi^2})^{\vee}(x)=\frac{1}{\sqrt{\pi}}e^{-x^2}$

$(e^{-4\pi^2\xi^2 t})^{\vee}(x)=\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$

$n\geq 1$时，$(e^{-4\pi^2|\xi|^2 t})^{\vee}(x)=\prod_{j=1}^{n}(e^{-4\pi^2\xi_j^2 t})^{\vee}(x_j)=\prod_{j=1}^{n}\frac{1}{2\sqrt{\pi t}}e^{-\frac{x_j^2}{4t}}=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$

回到热传导方程的解.

对方程 $\begin{cases} u_t-\Delta u=0 & x\in\mathbb{R}^n,\ t>0 \\ u|_{t=0}=\varphi(x) & x\in\mathbb{R}^n \end{cases}$

两边关于x作 Fourier 变换，则 $\begin{cases} \partial_t\hat{u}(\xi)+4\pi^2|\xi|^2\hat{u}(\xi)=0 \\ \hat{u}(\xi)|_{t=0}=\hat{\varphi}(\xi) \end{cases}$

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$$\Rightarrow \hat{u}(\xi,t)=\hat{\varphi}(\xi)e^{-4\pi^2|\xi|^2t}$$

取逆变换，由性质5 $f*g=(\hat{f}\hat{g})^\vee=\hat{f}^\vee*\hat{g}^\vee$（相乘的逆变换=逆变换的卷积）

$$\Rightarrow u(x,t)=(e^{-4\pi^2|\xi|^2t})^\vee*\varphi$$

$$=\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}*\varphi$$

$$=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}\varphi(y)dy$$

貌似在 $t=0$ 有奇性，下说明 $t=0$ 时满足初值

令 $k(x)=\frac{1}{(4\pi)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4}}$ （Heat kernel）

$k_t(x)=t^{-\frac{n}{2}}k(t^{-\frac{1}{2}}x),\ t>0$

则 $u(x,t)=\int_{\mathbb{R}^n}k_t(x-y)\varphi(y)dy=\int_{\mathbb{R}^n}k_t(y)\varphi(x-y)dy$

$\{k_t\}_{t>0}$ 有：(i) $\int_{\mathbb{R}^n}k_t(x)dx=\int_{\mathbb{R}^n}k(x)dx=1$

左 $=\int_{\mathbb{R}^n}t^{-\frac{n}{2}}k(t^{-\frac{1}{2}}x)dx=\int_{\mathbb{R}^n}k(y)dy=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x|^2}{4}}dx=1$

$(\int_{\mathbb{R}}e^{-x^2}dx=\sqrt{\pi})$

(ii) $\int_{\mathbb{R}^n}|k_t(x)|dx=1$

(iii) $\forall\eta>0,\ \int_{|x|>\eta}k_t(x)dx=\int_{|y|>\frac{\eta}{t^{\frac{1}{2}}}}k(y)dy\to0\ (t\to0^+)$

$\{k_t\}_{t>0}$ 为一族逼近恒等

$u(x,t)=k_t*\varphi\xrightarrow{t\to0^+}\varphi\qquad\varphi\in C(\mathbb{R}^n)$，有界

$$|u(x,t)-\varphi(x)|=\left|\int_{\mathbb{R}^n}k_t(y)\varphi(x-y)dy-\int_{\mathbb{R}^n}k_t(y)\varphi(x)dy\right|$$

$$=\left|\int_{\mathbb{R}^n}k_t(y)(\varphi(x-y)-\varphi(x))dy\right|$$

$$\le\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|y|^2}{4t}}|\varphi(x-y)-\varphi(x)|dy$$

$$\underset{z=\frac{y}{\sqrt{t}}}{=}\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|z|^2}{4}}|\varphi(x-z\sqrt{t})-\varphi(x)|dz$$

$$①=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|\geq R}e^{-\frac{|z|^2}{4}}\left|\varphi(x-\sqrt{t}z)-\varphi(x)\right|dz\leq\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|\geq R}e^{-\frac{|z|^2}{4}}dz\cdot 2\|\varphi\|_{L^\infty}\to 0\ (R\text{充分大})$$

$$②=\frac{1}{(4\pi)^{\frac{n}{2}}}\int_{|z|<R}e^{-\frac{|z|^2}{4}}\left|\varphi(x-\sqrt{t}z)-\varphi(x)\right|dz$$

由于φ连续，有 $\forall\ |z|<R$，$|\varphi(x-z\sqrt{t})-\varphi(x)|<\varepsilon$，$t\to 0^+$

则 ② $\leq C\varepsilon \Rightarrow |u(x,t)-\varphi(x)|\to 0$，$t\to 0^+$

Rmk. 1. 由K的光滑性以及 $u(x,t)=K_t(x)*\varphi$ 表明u具有光滑性

2. $\sup\limits_x |u(x,t)|\leq \sup\limits_x|\varphi(x)|$

由于 $u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|y|^2}{4t}}\varphi(x-y)dy$

$$\sup_x|u(x,t)|\leq\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|y|^2}{4t}}\sup_x|\varphi(x)|dy=\sup_x|\varphi(x)|$$

即温度最大值小于初始最大值，同时t足够大时 $u(x,t)\to 0$（衰减）

3. 热方程沿时间不反向演化，即从末态无法推及初态

4. 无限传播速度，$\forall$ 位置x的 $u(x,t)>0$

考虑非齐次方程

对方程 $\begin{cases}u_t-\Delta u=f(x,t) & x\in\mathbb{R}^n,\ t>0\\ u(x,0)=\varphi(x) & x\in\mathbb{R}^n\end{cases}$

取Fourier变换，有 $\begin{cases}\partial_t\hat{u}+4\pi^2|\xi|^2\hat{u}=\hat{f}(\xi,t)\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi)\end{cases}$

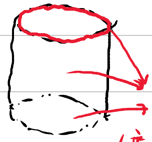
$$\Rightarrow \hat{u}(\xi,t)=\hat{\varphi}(\xi)e^{-4\pi^2|\xi|^2t}+\int_0^t e^{-4\pi^2|\xi|^2(t-s)}\hat{f}(\xi,s)ds$$

取Fourier逆变换，有

$$u(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4t}}\varphi(y)dy+\int_0^t\frac{1}{(4\pi(t-s))^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4(t-s)}}f(y,s)dyds$$

§3.2 极值原理和最大模估计

考虑
$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega\times\{t>0\} \\ u|_{t=0} = \varphi(x) & \text{in } \Omega \\ u(x,t) = h(x,t) & \text{on } \partial\Omega\times\{t>0\} \end{cases}$$



令 $Q_T = \Omega\times(0,T]$，定义抛物边界 $\Gamma_T = \overline{Q}_T \setminus Q_T$

thm1. 极值原理

$u\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 满足 $\mathcal{L}u \doteq u_t - u_{xx} = f \le 0$

则 $u(x,t)$ 在 $\overline{Q}_T$ 上最大值必在 Γ_T 上达到，即

$$\max_{\overline{Q}_T} u(x,t) = \max_{\Gamma_T} u(x,t)$$

pr: 令 $M = \max_{\overline{Q}_T} u(x,t)$，$m = \max_{\Gamma_T} u(x,t)$

step 1. $f<0$，若 $M>m$，

则 f 在 $(x_0,t_0)\in Q_T$ 上达到最大值

则 $u_{xx}(x_0,t_0)\le 0$，$u_t(x_0,t_0)\ge 0$

$\mathcal{L}u(x_0,t_0) = u_t(x_0,t_0) - u_{xx}(x_0,t_0) \ge 0$ 矛盾

step 2. $f\le 0$，令 $v = u-\varepsilon t$，则 $\mathcal{L}v = \mathcal{L}u - \varepsilon = f-\varepsilon < 0$

由 step 1.

$$\max_{\overline{Q}_T} u - \varepsilon T \le \max_{\overline{Q}_T} v = \max_{\Gamma_T} v \le \max_{\Gamma_T} u$$

由 ε 的任意性，$\max_{\overline{Q}_T} u = \max_{\Gamma_T} u$

Rmk. 则 $\mathcal{L}u = f \ge 0$，则 u 在 $\overline{Q}_T$ 上最小值必在边界处取到

thm 2. 比较原理

$u, v\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 满足 $\mathcal{L}u \le \mathcal{L}v$，$u|_{\partial Q_T} \le v|_{\partial Q_T}$，则在 $\overline{Q}_T$，$u(x,t)\le v(x,t)$

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pr: 令 $w(x,t)=u(x,t)-v(x,t)$

$\mathcal{L}w=\mathcal{L}u-\mathcal{L}v\le 0$, $w|_{\partial Q_T}\le 0$

故由极值原理, $\max\limits_{\overline{Q}_T} w=\max\limits_{\partial Q_T} w$

则 $u(x,t)\le v(x,t)$ $\forall (x,t)\in Q_T$

thm 3. 最大模估计

设 $u\in C^{2,1}(Q_T)\cap C(\overline{Q}_T)$ 为方程 $\begin{cases}\mathcal{L}u=f, & x\in(0,l)\times\{t>0\}\\ u|_{t=0}=\varphi(x) & x\in[0,l]\\ u|_{x=0}=g_1(t) & u|_{x=l}=g_2(t)\end{cases}$ 的解, 则

$\max\limits_{\overline{Q}_T}|u(x,t)|\le FT+B$

$F=\max\limits_{Q_T}|f|$, $B=\max\left\{\max\limits_{x\in[0,l]}|\varphi(x)|, \max\limits_{[0,T]}|g_1(t)|, \max\limits_{[0,T]}|g_2(t)|\right\}$

pr: $v=Ft+B-u$

$\begin{cases}\mathcal{L}v=F-f\ge 0\\ v|_{\partial Q_T}=Ft+B-g_1/g_2\ge 0\end{cases}$

由极大值原理, $\min\limits_{\overline{Q}_T} v=\min\limits_{\partial Q_T} v\ge 0$

$\Rightarrow v(x,t)\ge 0$ $\forall (x,t)\in\overline{Q}_T$

则 $u(x,t)\le Ft+B$

对 $v=Ft+B+u$ 完成相同过程 $\Rightarrow -u(x,t)\le Ft+B$

$\Rightarrow |u(x,t)|\le Ft+B\le FT+B$

Rmk. 可证明热方程解的唯一性、稳定性

下考虑其余两类边值问题解的唯一性与稳定性

第二类边值问题

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$$\begin{cases} u_t-u_{xx}=f(x,t) \\ u(0,t)=\mu_1(t) \quad u_x+hu\,(l,t)=\mu_2(t) \quad h>0 \\ u(x,0)=\varphi(x) \end{cases}$$

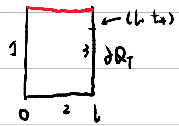
pr: 只需证明 $\begin{cases} u_t-u_{xx}=0 \\ u(0,t)=0 \quad u_x+hu\,(l,t)=0 \quad h>0 \\ u(x,0)=0 \end{cases}$

只有零解.

否则在 Q_T 上有非零解，其有正的最大值或负的最小值.

$\mathcal{L}u=0$. 由极值原理，最大值最小值均在边界处取到

不妨设正的最大值在边界取到



$u(0,t)=u(x,0)=0 \Rightarrow$ 正的最大值在3取到

设u在 (l,t_*) 取到正的最大值

$u_x(l,t_*)\geq 0$，故 $u_x+hu\,(l,t_*)>0$ 与边条件矛盾

进而给出了第三类边值下解的唯一性

第二类边值问题

$$\begin{cases} u_t-u_{xx}=f(x,t) \\ u(0,t)=\mu_1(t), \quad u_x(l,t)=\mu_2(t) \\ u(x,0)=\varphi(x) \end{cases}$$

只需证明 $\begin{cases} u_t-u_{xx}=0 \\ u(0,t)=0 \quad u_x(l,t)=0 \\ u(x,0)=0 \end{cases}$

只有零解.

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pr: idea $(uw)_x = u_x w + u w_x$

将第二类边值转化为第三类边值

令 $\tilde{u}(x,t) = w(x)u(x,t)$, $u = \frac{\tilde{u}}{w}$

则 $u_t = \frac{\tilde{u}_t}{w}$, $u_x = \frac{\tilde{u}_x w - w_x \tilde{u}}{w^2}$, $u_{xx} = \left(-\frac{w_{xx}}{w^2} + 2\frac{w_x^2}{w^3}\right)\tilde{u} - 2\frac{w_x}{w^2}\tilde{u}_x + \frac{1}{w}\tilde{u}_{xx}$

$\Rightarrow \frac{\tilde{u}_t}{w} + \left(\frac{w_{xx}}{w^2} - 2\frac{w_x^2}{w^3}\right)\tilde{u} + \frac{2w_x}{w^2}\tilde{u}_x - \frac{1}{w}\tilde{u}_{xx}$

$$\Rightarrow \begin{cases} \tilde{u}_t - \tilde{u}_{xx} = -2\frac{w_x}{w}\tilde{u}_x - \left(\frac{w_{xx}}{w} - 2\frac{w_x^2}{w^2}\right)\tilde{u} \\ \tilde{u}(0,t) = 0, \quad \tilde{u}_x(l,t) = w_x u + u_x w\,(l,t) = w_x(l)\,u(l,t) = w_x(l)\cdot\frac{\tilde{u}(l,t)}{w(l)} \\ \tilde{u}(x,0) = 0 \end{cases}$$

hope $\frac{w_x(l)}{w(l)} = -1$, 取 $w(x) = -x+l+1$

$$\text{则} \begin{cases} \tilde{u}_t - \tilde{u}_{xx} = \frac{2}{-x+l+1}\tilde{u}_x + 2\frac{1}{(-x+l+1)^2}\tilde{u} \quad (*) \\ \tilde{u}(0,t) = 0, \quad (\tilde{u}_x + \tilde{u})(l,t) = 0 \\ \tilde{u}(x,0) = 0 \end{cases}$$

令 $v(x,t) = e^{-\lambda t}\tilde{u}(x,t)$, $\tilde{u} = e^{\lambda t}v$

$\tilde{u}_t = v_t e^{\lambda t} + \lambda e^{\lambda t} v$

$\tilde{u}_x = e^{\lambda t} v_x \quad \tilde{u}_{xx} = e^{\lambda t} v_{xx}$

$$\text{则} \begin{cases} v_t - v_{xx} - \frac{2}{-x+l+1}v_x + \left(\lambda - \frac{2}{(-x+l+1)^2}\right)v = 0 \\ v(0,t) = 0, \quad (v_x + v)(l,t) = 0 \\ v(x,0) = 0 \end{cases}$$

取 $\lambda > 2$, 则 $\lambda - \frac{2}{(-x+l+1)^2} > 0$

claim: v 在 Q_T 上 ? 的最大值在边界取到

事实上, 若在 $v(x_0, t_0) = M > 0$, $(x_0, t_0) \in Q_T$

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则 $V_t(x_0, t_0) \geq 0$，$V_x(x_0, t_0) = 0$，$V_{xx}(x_0, t_0) \leq 0$

与方程条件矛盾

再利用第三类问题的处理给出第三类问题解的唯一性

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§3.3 初值问题的最大模估计

考虑实值函数的初值问题

$$\begin{cases} u_t - u_{xx} = f(x,t) & x\in R,\ 0<t\leq T \\ u(x,0) = \varphi(x) & x\in R \end{cases}$$

thm 1. 最大模估计

假设 $u\in C^{2,1}(Q_T)\cap C(\overline{Q_T})$ 为上述问题的有界解

则 $\sup\limits_{Q_T}|u(x,t)| \leq T\sup\limits_{Q_T}|f(x,t)| + \sup\limits_{x\in R}|\varphi(x)|$

令 $F=\sup\limits_{Q_T}|f(x,t)|$, $\Phi=\sup\limits_{x\in R}|\varphi(x)|$, $M=\sup\limits_{Q_T}|u(x,t)|$

pr: $\forall L>0$, 在 $Q_T^L=(-L,L)\times(0,T]$ 上考虑辅助函数

$$w(x,t) = Ft+\Phi+V_L(x,t)\pm u(x,t)$$

$$V_L(x,t) = \frac{M}{L^2}(x^2+2t)$$

(idea: $V_L(x,t)$ 为自由方程 $V_t-V_{xx}=0$ 的解)

则 $w_t - w_{xx} = F\pm f \geq 0$

由极值原理, 最小值在边界处取到

$$w\big|_{t=0} = \Phi+\frac{M}{L^2}x^2\pm\varphi(x)\geq 0$$

$$w\big|_{x=\pm L} = Ft+\Phi+M\pm u\geq 0$$

$\Rightarrow w(x,t)\geq 0 \quad (x,t)\in Q_T^L$

$\forall (x_0,t_0)\in Q_T$, 取 L 充分大, st. $(x_0,t_0)\in Q_T^L$

由于 $w(x_0,t_0)\geq 0$, 故

$$|u(x_0,t_0)| \leq Ft_0+\Phi+\frac{M}{L^2}(x_0^2+2t_0)$$

$$\leq FT+\Phi+\frac{M}{L^2}(x_0^2+2t_0)$$

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令 $L \to +\infty$. 则 $|u(x_0,t_0)| \le FT + \Phi$

由 (x_0,t_0) 的任意性, $\sup\limits_{Q_T} |u(x,t)| \le T \sup\limits_{Q_T} |f(x,t)| + \sup\limits_{x\in\mathbb{R}} |\varphi(x)|$

Rmk. 有界性可以放宽为 $|u(x,t)| \le Me^{A|x|^2}$, $(x,t) \in Q_T$

下面用能量估计证明解的唯一性

考虑方程

$$\begin{cases} u_t - u_{xx} = f & (x,t) \in Q_T = (0,l) \times (0,T] \\ u|_{t=0} = \varphi(x) \\ u|_{x=0} = u|_{x=l} = 0 \end{cases}$$

thm 2. 能量不等式

设 $u \in C^{1,0}(\overline{Q}_T) \cap C^{2,1}(Q_T)$ 为上述问题的解,

则 $$\sup_{0\le t\le T} \int_0^l u^2(x,t)\,dx + 2\int_0^T\int_0^l u_x^2(x,t)\,dx\,dt$$
$$\le M\left(\int_0^l \varphi^2(x)\,dx + \int_0^T\int_0^l f^2(x,t)\,dx\,dt\right)$$

pr: $(u_t - u_{xx})u = fu$

左 $= \frac{1}{2}(u^2)_t - (uu_x)_x + (u_x)^2 = fu$

在 $[0,l]$ 上积分, 有

$$\frac{d}{dt}\frac{1}{2}\int_0^l u^2\,dx - uu_x\Big|_{x=0}^{x=l} + \int_0^l u_x^2\,dx = \int_0^l fu\,dx$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2\,dx\right) + \int_0^l u_x^2\,dx \le \frac{1}{2}\int_0^l f^2\,dx + \frac{1}{2}\int_0^l u^2\,dx$$

$$\Rightarrow \frac{d}{dt}\left(\int_0^l u^2\,dx\right) \le \int_0^l f^2\,dx + \int_0^l u^2\,dx$$

$$\frac{d}{dt}\left(e^{-t}\int_0^l u^2\,dx\right) \le e^{-t}\int_0^l f^2\,dx$$

从 $(0,t)$ 积分 $\Rightarrow$ $e^{-t}\int_0^l u^2\,dx - y(0) \le \int_0^t e^{-s}\int_0^l f^2(x,s)\,dx\,ds$

$$\stackrel{def}{} \quad y(t) = \int_0^l u^2(x,t)\,dx$$

则 $y(t)\le e^t y(0)+e^t\int_0^t\int_0^l f^2(x,s)\,dxds$

$$\Rightarrow \int_0^l u^2dx \le e^t\left(\int_0^l \varphi^2dx+\int_0^t\int_0^l f^2(x,s)\,dxds\right)$$

$$\le e^T\left(\int_0^l \varphi^2dx+\int_0^T\int_0^l f^2(x,s)\,dxds\right) \quad \forall\, 0<t\le T$$

故 $\frac{d}{dt}\left(\frac{1}{2}\int_0^l u^2dx\right)+\int_0^l u_x^2dx \le \frac{1}{2}\int_0^l f^2dx+\frac{1}{2}e^t\left(\int_0^l \varphi^2dx+\int_0^T\int_0^l f^2(x,s)\,dxds\right)$

再从 0 到 t 积分，有

$$\frac{1}{2}\int_0^l u^2dx+\int_0^t\int_0^l u_x^2dxdt \le \frac{1}{2}\int_0^l \varphi^2dx+\frac{1}{2}\int_0^t\int_0^l f^2dxdt$$

$$+\frac{1}{2}(e^t-1)\left[\int_0^l \varphi^2dx+\int_0^T\int_0^l f^2(x,s)\,dxds\right]$$

$$\le \frac{1}{2}e^T\int_0^T\int_0^l f^2(x,s)\,dxds \quad \forall\, 0<t\le T$$

$$\Rightarrow \sup_{0\le t\le T}\int_0^l u^2(x,t)\,dx+2\int_0^T\int_0^l u_x^2(x,t)\,dxdt$$

$$\le e^T\left(\int_0^l \varphi^2(x)\,dx+\int_0^T\int_0^l f^2(x,t)\,dxdt\right)$$

Rmk. 可得到解的唯一性与稳定性.

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第四章 波动方程

方程的来源：拉弦 细弦/薄膜/弹性体的自由/受迫振动

$u(x,t)$ 未知 $\quad \Delta u = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 u$

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega,\ t>0 \\ u(x,0) = f(x) & x\in\Omega \\ u_t(x,0) = g(x) & x\in\Omega \\ \text{边值} \end{cases}$$

初值，为关于 x 的函数

Rmk. 物理意义

$f(x,t)$ 表示单位质量所受外力

第一类边值 (Dirichlet) $u(x,t) = h(x,t) \quad \forall x\in\partial\Omega$

第二类边值 (Newmann) $\frac{\partial u}{\partial \vec{n}}(x,t) = h(x,t) \quad \forall x\in\partial\Omega$

第三类边值 (Robin) $\frac{\partial u}{\partial \vec{n}}(x,t) + \alpha(x,t)\, u(x,t) = h(x,t) \quad \forall x\in\partial\Omega \quad \alpha(x,t)>0$

Rmk. 边值意义

(Dirichlet) 边界点位移变化

若 $h(x,t) \equiv h(x)$，则边界点固定

(Newmann) 边界点受力情况

若 $h(x,t) \equiv 0$，则无外力通过 $\partial\Omega$ 对弹性体作用

(Robin) 位移与受力线性组合

若 $h(x,t) \equiv 0$，则 $\partial\Omega$ 固定在支架

§4.1 初值问题

§4.1.1 一阶偏微分方程的解

考虑: $\begin{cases} \dfrac{\partial u}{\partial t} + a(x,t)\dfrac{\partial u}{\partial x} + b(x,t)u = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$

$u(x,t)$ 为未知函数 $-\infty<x<+\infty$, $t>0$

若 $x=x(t)$, 令 $u(x(t),t) = U(t)$

$$\frac{dU}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} x'(t)$$

$x(t)$ 称为特征线

若 $x'(t) = a(x(t),t)$, $\dfrac{dU}{dt} + b(x(t),t)U(t) = f(x(t),t)$

令 $x(0)=c$, 则 $U(0) = u(x(0),0) = \phi(c)$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = a(x(t),t) \\ x(0) = c \end{cases} \qquad \begin{cases} \dfrac{dU}{dt} + b(x(t),t)U(t) = f(x(t),t) \\ U(0) = \phi(c) \end{cases}$$

转化为2个ode

ex. $\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = \phi(x) \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = -a \\ x(0) = c \end{cases} \Rightarrow x(t) = -at + c$$

$$\begin{cases} \dfrac{dU}{dt} = 0 \\ U(0) = \phi(c) \end{cases} \Rightarrow U = \phi(c)$$

$U(t) = \phi(x(t)+at) = u(x(t),t)$.

取 $u(x,t) = \phi(x+at)$.

ex. $\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial u}{\partial x} = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = -a \\ x(0) = c \end{cases} \Rightarrow x(t) = -at + c$$

$$\begin{cases} \dfrac{dU}{dt} = f(x(t),t) \\ U(0) = \phi(c) \end{cases}$$

$x(t)$ 带入, $\dfrac{dU}{dt} = f(-at+c,t)$

$$\Rightarrow U(t) = \phi(c) + \int_0^t f(-a\tau+c,\tau)d\tau$$

$$u(x,t) = \phi(x+at) + \int_0^t f(x+a(t-\tau),\tau)d\tau$$

ex. $\begin{cases} \frac{du}{dt} + (x+t)\frac{du}{dx} + u = x \\ u|_{t=0} = x \end{cases}$

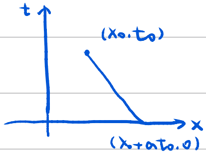
$\begin{cases} \frac{dx}{dt} = x+t \\ x(0) = C \end{cases}$ (一阶线性) $\Rightarrow x(t) = Ce^{t} - t - 1$

$\begin{cases} \frac{dU}{dt} + U(t) = Ce^{t} + e^{t} - t - 1 \\ U(0) = C \end{cases}$ $\Rightarrow U(t) = -t + \frac{1}{2}(e^{t} - e^{-t}) + \frac{C}{2}(e^{t} - e^{-t})$

$\Rightarrow u(x,t) = \frac{1}{2}(x-t+1) - e^{-t} + \frac{1}{2}(x+t+1)e^{-2t}$

Rmk. 对第一个方程有如下较为几何的解释

在 $x(t) = -at + C$ 上，有 $\frac{dU}{dt} = 0$



依据初值 $u(x_0+at_0, 0) = \phi(x_0+at_0) = u(x_0,t_0) \quad \forall x_0, t_0$

$\Rightarrow u(x,t) = \phi(x+at)$

同样地，对第二个方程

$u(x_0+at_0, 0) = \phi(x_0+at_0)$

$u(x_0,t_0) = u(x_0+at_0,0) + \int_0^{t_0} f(x(t),t)\,dt$

$= \phi(x_0+at_0) + \int_0^{t_0} f(x_0+a(t_0-t),t)\,dt \quad \forall x_0, t_0$

$\Rightarrow u(x,t) = \phi(x+at) + \int_0^{t} f(x+a(t-\tau),\tau)\,d\tau$

特征线法将PDE转化为ODE，最终结果即在特征线上对时间积分

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§4.1.2 问题的简化

对 R^n，仅需提出初值条件（无边界）

考察初值问题

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x) \qquad x\in R^n, t>0 \qquad (4.6) \\ u_t(x,0) = \psi(x) \end{cases}$$

用构造的办法给出初值问题的解，暂未说明唯一性

令 u_1 满足 $\begin{cases} \partial_t^2 u_1 - \Delta u_1 = 0 \\ u_1(x,0) = \varphi(x) \qquad (4.7) \\ \partial_t u_1(x,0) = 0 \end{cases}$ "位移"

令 u_2 满足 $\begin{cases} \partial_t^2 u_2 - \Delta u_2 = 0 \\ u_2(x,0) = 0 \qquad (4.8) \\ \partial_t u_2(x,0) = \Psi(x) \end{cases}$ "速度"

令 u_3 满足 $\begin{cases} \partial_t^2 u_3 - \Delta u_3 = f(x,t) \\ u_3(x,0) = 0 \qquad (4.9) \\ \partial_t u_3(x,0) = 0 \end{cases}$ "外力"

则 $u_1+u_2+u_3$ 为 (4.6) 的解

thm 1. $u_2 = M_\psi(x,t)$ 为 (4.8) 解，则 (4.7)、(4.9) 的解为

$$u_1 = \frac{\partial}{\partial t} M_\varphi, \quad u_3 = \int_0^t M_{f_\tau}(x, t-\tau)\,d\tau$$

pr. 令 $\tilde{u}_1 = M_\varphi$，则 $\begin{cases} \partial_t^2 \tilde{u}_1 - \Delta \tilde{u}_1 = 0 \\ \tilde{u}_1(x,0) = 0 \\ \partial_t \tilde{u}_1(x,0) = \varphi(x) \end{cases}$

令 $v = \partial_t \tilde{u}_1$，则 $\begin{cases} \partial_t^2 v - \Delta v = 0 \\ v(x,0) = \varphi(x) \\ \partial_t v(x,0) = \partial_t^2 \tilde{u}_1(x,0) = \Delta \tilde{u}_1(x,0) = 0 \end{cases}$

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Rmk. 初值为先拾出函数，如 Δu，u_t，u_{tt}，再赋值 先赋值，再拾出函数

在上述过程中 $v(x,0)=\hat{u}_t(x,0)=\frac{\partial}{\partial t}\hat{u}(x,0)\times\frac{\partial}{\partial t}0=0$

对其余变量求偏导时，可交换拾出函数与赋值顺序

令 $\hat{u}_1(x,t)=M_{f_\tau}(x,t)$，则 $\begin{cases}\partial_t^2\hat{u}_1-\Delta\hat{u}_1=0\\ \hat{u}_1(x,0)=0\\ \partial_t\hat{u}_1(x,0)=f_\tau=f(x,\tau)\end{cases}$ τ 为参数

令 $v(x,t)=M_{f_\tau}(x,t-\tau)=\hat{u}_1(x,t-\tau)$，则

$$\begin{cases}\partial_t^2 v-\Delta v=(\partial_t^2\hat{u}_1-\Delta\hat{u}_1)(x,t-\tau)=0\\ v\big|_{t=\tau}=\hat{u}_1(x,0)=0\\ \partial_t v\big|_{t=\tau}=\partial_t\hat{u}_1(x,0)=f(x,\tau)\end{cases}$$

$$u_3=\int_0^t M_{f_\tau}(x,t-\tau)d\tau$$

$$\partial_t u_3=M_{f_t}(x,0)+\int_0^t\frac{\partial}{\partial t}M_{f_\tau}(x,t-\tau)d\tau$$

$$=\int_0^t\frac{\partial}{\partial t}M_{f_\tau}(x,t-\tau)d\tau$$

$$\partial_t^2 u_3=\frac{\partial}{\partial t}M_{f_t}(x,0)+\int_0^t\frac{\partial^2}{\partial t^2}M_{f_\tau}(x,t-\tau)d\tau$$

$$=f(x,t)+\int_0^t\Delta M_{f_\tau}(x,t-\tau)d\tau$$

$$=f(x,t)+\Delta\int_0^t M_{f_\tau}(x,t-\tau)d\tau$$

$$=f(x,t)+\Delta u_3$$

且 $u_3(x,0)=0$，$\partial_t u_3(x,0)=0$

Rmk. "Δ" 可拿出来，由于为对 x 的导数，积分限不含 x

该过程被称作冲量原理 (Duhamel) 非齐次方程 → 具有初速度齐次方程解的和

$$u_3(x,t)=\lim_{\|\Delta\|\to 0}\sum_{i=0}^{n-1}M_{f_{t_i}}(x,t-t_i)\Delta t_i=\lim_{\|\Delta\|\to 0}\sum_{i=0}^{n-1}M_{f_{t_i}\Delta t_i}(x,t-t_i)$$

故问题转化为如下考查方程 $\begin{cases}\partial_t^2 u-\Delta u=0\\ u(x,0)=0,\ \partial_t u(x,0)=\psi(x)\end{cases}$

的解，thm 1. 指出了"速度"方程具有本质性.

Recall: 傅立叶变换 $\hat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ ξ 为参向量

$$\widehat{\Delta f}(\xi)=\int_{\mathbb{R}^n}\Delta f(x)e^{-2\pi i x\cdot\xi}dx=\sum_j\int_{\mathbb{R}^n} f_{x_jx_j}e^{-2\pi i x\cdot\xi}dx$$

$$\overset{\uparrow}{=}-\sum_j\int_{\mathbb{R}^n} f_{x_j}\cdot(-2\pi i\xi_j)e^{-2\pi i x\cdot\xi}dx$$

f, f_{x_i} 在无穷远处 $\to 0$

$$=-\sum_j(-2\pi i\xi_j)\int_{\mathbb{R}^n} f_{x_j}e^{-2\pi i x\cdot\xi}dx$$

$$=\sum_j(-2\pi i\xi_j)^2\int_{\mathbb{R}^n} f e^{-2\pi i x\cdot\xi}dx$$

$$=-4\pi^2|\xi|^2\hat{f}(\xi)$$

将分析运算转化为代数运算

方程 $\begin{cases}\partial_t^2 u-\Delta u=0\\ u\big|_{t=0}=\varphi(x),\ u_t\big|_{t=0}=\psi(x)\end{cases}$

两边关于x做傅立叶变换，$\begin{cases}\partial_t^2\hat{u}(\xi,t)+4\pi^2|\xi|^2\hat{u}(\xi,t)=0\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi),\ \partial_t\hat{u}(\xi,0)=\hat{\psi}(\xi)\end{cases}$

ξ 视作参数，则为二阶常微分方程，$\lambda^2+4\pi^2|\xi|^2=0 \Rightarrow \lambda=\pm 2\pi|\xi|i$

$$\Rightarrow \hat{u}(\xi,t)=C_1\cos(2\pi t|\xi|)+C_2\sin(2\pi t|\xi|)$$

$$\hat{u}(\xi,0)=C_1=\hat{\varphi}(\xi)$$

$$\partial_t\hat{u}(\xi,0)=C_2\cdot 2\pi|\xi|=\hat{\psi}(\xi)$$

$$\Rightarrow \hat{u}(\xi,t)=\cos(2\pi t|\xi|)\hat{\varphi}(\xi)+\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}\hat{\psi}(\xi)$$

Rmk. 也说明了 u_1, u_2 解的关系

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方程 $\begin{cases} \partial_t^2 u_3 - \Delta u_3 = f(x,t) \\ u_3(x,0)=0,\ \partial_t u_3(x,0)=0 \end{cases}$

关于x做傅立叶变换, $\begin{cases} \partial_t^2 \hat{u}_3 + 4\pi^2|\xi|^2\hat{u}_3 = \hat{f}(\xi,t) \\ \hat{u}_3(\xi,0)=0,\ \partial_t\hat{u}_3(\xi,0)=0 \end{cases}$

$\Rightarrow \hat{u}_3(\xi,t) = \int_0^t \frac{\sin(2\pi|\xi|(t-\tau))}{2\pi|\xi|}\hat{f}(\xi,\tau)\,d\tau$

Rmk. 也说明了 u_2, u_3 的关系.

若对x.t同时变换　$x\to\xi,\ t\to s$

$-4\pi^2 s^2\hat{u} + 4\pi^2|\eta|^2\hat{u} = 0$

$\Rightarrow (|\eta|^2 - s^2)\,\hat{u}(s,\xi) = 0$

$\Rightarrow s^2 = |\eta|^2$　$(s,\eta)\in\mathbb{R}^{1+n}$ 构成锥面

表明 $\hat{u}$ 仅在锥面上可不为0, 波方程对时空的Fourier变换

支在锥面上.

§4.1.3 一维初值问题

考察 R上波动方程
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) & x\in\mathbb{R},\ t>0 \\ u(x,0) = \varphi(x) & x\in\mathbb{R} \\ \partial_t u(x,0) = \psi(x) & x\in\mathbb{R} \end{cases} \quad (4.13)$$

Rmk. X的范围会影响解方程的方法与结果

$$\begin{cases} \partial_t^2 u_2 - \Delta u_2 = 0 \\ u_2(x,0) = 0 \\ \partial_t u_2(x,0) = \Psi(x) \end{cases} \quad (4.8)$$ (由上节thm知只需求解该方程即得一般波动方程的解)

即 $(\partial_t + \partial_x)(\partial_t - \partial_x) u = 0$

令 $v(x,t) = (\partial_t - \partial_x) u$，则 $\begin{cases} \partial_t v + \partial_x v = 0 \\ v(x,0) = (\partial_t - \partial_x) u(x,0) = \Psi(x) \end{cases}$

$$\Rightarrow v(x,t) = \Psi(x-t)$$

$$\begin{cases} \partial_t u - \partial_x u = \Psi(x-t) \\ u(x,0) = 0 \end{cases}$$

$$\Rightarrow u(x,t) = \int_0^t \Psi(\underbrace{x+t-2\tau}_{\overset{\triangle}{=} y})\, d\tau$$

$$= \frac{1}{2}\int_{x-t}^{x+t} \Psi(y)\, dy$$

故 $u_2(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \Psi(y)\, dy$

$$u_1(x,t) = \frac{d}{dt}\left(\frac{1}{2}\int_{x-t}^{x+t} \varphi(y)\, dy\right)$$
$$= \frac{1}{2}\left(\varphi(x+t) + \varphi(x-t)\right)$$

$$u_3(x,t) = \int_0^t \frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\, dy\, d\tau$$
$$= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\, dy\, d\tau$$

故(4.13)解为

$$u_1 + u_2 + u_3 \quad (4.20)$$

thm 2 (D'Alembert 公式)

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若 $f\equiv 0$，令 $F(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}\int_0^x \psi(y)\,dy$

$G(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}\int_x^0 \psi(y)\,dy$

$\Rightarrow u(x,t)=F(x+t)+G(x-t)$

左行波　右行波

thm 3　给出形式解存在条件（形式解 → 古典解）

$\varphi\in C^2(\mathbb{R})$，$\psi\in C^1(\mathbb{R})$，$f\in C^1(\mathbb{R}\times\overline{\mathbb{R}^+})$

则 (4.20) 给出的函数 $u\in C^2(\mathbb{R}\times\overline{\mathbb{R}_+})$，且为初值问题 (4.13) 解

pr. $u(x,t)=\frac{1}{2}(\varphi(x+at)+\varphi(x-at))+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\,dy+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(y,\tau)\,dy\,d\tau$

① $u(x,t)\in C(\mathbb{R}\times\overline{\mathbb{R}_+})$　为连续函数复合 / 变限积分

② $u_t(x,t)=\frac{a}{2}\varphi'(x+at)-\frac{a}{2}\varphi'(x-at)+\frac{1}{2a}\left(a\psi(x+at)+a\psi(x-at)\right)$

$+\frac{1}{2a}\int_0^t af(x+a(t-\tau),\tau)+af(x-a(t-\tau),\tau)\,d\tau$

$u_x(x,t)=\frac{1}{2}\varphi'(x+at)+\frac{1}{2}\varphi'(x-at)+\frac{1}{2a}(\psi(x+at)-\psi(x-at))$

$+\frac{1}{2a}\int_0^t f(x+a(t-\tau),\tau)-f(x-a(t-\tau),\tau)\,d\tau$

$u(x,t)\in C^1(\mathbb{R}\times\overline{\mathbb{R}_+})$，由于 u_t、$u_x\in C(\mathbb{R}\times\overline{\mathbb{R}_+})$

③ $u_{tt}(x,t)=\frac{a^2}{2}\varphi''(x+at)+\frac{a^2}{2}\varphi''(x-at)+\frac{1}{2a}(a^2\psi'(x+at)-a^2\psi'(x-at))$

$+\frac{1}{2a}\left[2af(x,t)+\int_0^t a^2f_x(x+a(t-\tau),\tau)-a^2f_x(x-a(t-\tau),\tau)\,d\tau\right]$

$u_{xx}(x,t)=\frac{1}{2}\varphi''(x+at)+\frac{1}{2}\varphi''(x-at)+\frac{1}{2a}(\psi'(x+at)-\psi'(x-at))$

$+\frac{1}{2a}\int_0^t\left[f_x(x+a(t-\tau),\tau)-f_x(x-a(t-\tau),\tau)\right]d\tau$

$u_{xt}(x,t)=\frac{a}{2}\varphi''(x+at)-\frac{a}{2}\varphi''(x-at)+\frac{1}{2}(\psi'(x+at)+\psi'(x-at))$

$+\frac{1}{2a}\int_0^t af_x(x+a(t-\tau),\tau)+af_x(x-a(t-\tau),\tau)\,d\tau$

$\in C(\mathbb{R}\times\overline{\mathbb{R}_+})\quad=u_{tx}(x,t)$

$u(x,t) \in C^2(\mathbb{R}\times\overline{\mathbb{R}_+})$

④ $u_{tt} - a^2 u_{xx} = f(x,t)$

$u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$

则 u 为初值问题的解

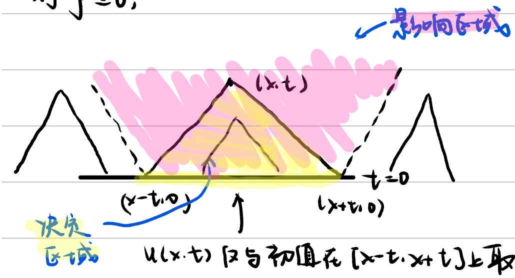
thm 4 利用形式解分析性质

若 φ、ψ、f 为 x 的偶/奇/周期为 l 函数

由 (4.20) 给出的解 u 为 x 的偶/奇/周期为 l 函数

Rmk. 几何性质

对 $f \equiv 0$,



$u(x,t)$ 只与初值在 $[x-t, x+t]$ 上取值有关

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§4.1.4 一维半无界问题

考虑波动方程
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) & x>0 \quad t>0 \\ u(x,0)=\varphi(x) & x\geq 0 \\ \partial_t u(x,0) = \psi(x) & x\geq 0 \\ u(0,t)=g(t) & t\geq 0 \quad (\text{边值}) \end{cases} \qquad (4.21)$$

1、若 $g(t)\equiv 0$，作奇延拓，

令 $\bar{\varphi}(x)=\begin{cases}\varphi(x) & x\geq 0\\ -\varphi(-x) & x<0\end{cases}$ $\quad \bar{\psi}(x)=\begin{cases}\psi(x) & x\geq 0\\ -\psi(-x) & x<0\end{cases}$ $\quad \bar{f}(x,t)=\begin{cases}f(x,t) & x\geq 0\\ -f(-x,t) & x<0\end{cases}$

Rmk. 奇延拓，则 $w(0)=0$；偶延拓，则 $w'(0)=0$

令 $\bar{u}(x,t)$ 为方程
$$\begin{cases}\partial_t^2\bar{u}-\partial_x^2\bar{u}=\bar{f}(x,t) & x\in\mathbb{R} \quad t>0\\ \bar{u}(x,0)=\bar{\varphi}(x) & x\in\mathbb{R}\\ \partial_t\bar{u}(x,0)=\bar{\psi}(x) & x\in\mathbb{R}\end{cases}$$

的解，由于对 x 的奇性，$\bar{u}(0,t)=0$，$t\geq 0$ 满足边值

由 (4.20)，$\bar{u}(x,t)=\frac{1}{2}(\bar{\varphi}(x+t)+\bar{\varphi}(x-t))+\frac{1}{2}\int_{x-t}^{x+t}\bar{\psi}(y)dy+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\bar{f}(y,\tau)\,dyd\tau$ $\quad x\in\mathbb{R}, t>0$

$x>0$，$x\geq t$ 时，

$$u(x,t)=\frac{1}{2}(\varphi(x+t)+\varphi(x-t))+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dyd\tau \qquad (4.23)$$

$x>0$，$x<t$ 时，

$$u(x,t)=\frac{1}{2}(\varphi(x+t)-\varphi(t-x))+\frac{1}{2}\int_0^{x+t}\psi(y)dy+\frac{1}{2}\int_{x-t}^{0}-\psi(-y)dy$$
$$+\frac{1}{2}\int_{t-x}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau+\frac{1}{2}\int_0^{t-x}\int_0^{x+(t-\tau)}f(y,\tau)dyd\tau+\frac{1}{2}\int_0^{t-x}\int_{x-(t-\tau)}^{0}-f(-y,\tau)dyd\tau$$
$$=\frac{1}{2}(\varphi(x+t)-\varphi(t-x))+\frac{1}{2}\int_{t-x}^{t+x}\psi(y)dy$$
$$+\frac{1}{2}\int_{t-x}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau+\frac{1}{2}\int_0^{t-x}\int_{(t-\tau)-x}^{(t-\tau)+x}f(y,\tau)dyd\tau \qquad (4.24)$$

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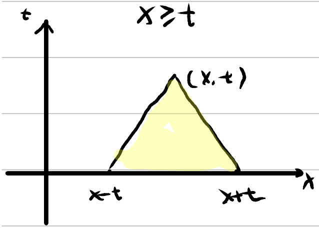

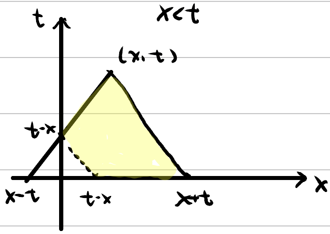

黄色部分所示为积分区域

如何完成从形式解到古典解过度？ Rmk. 古典解：所需的各阶偏导数连续可微，且在边界处连续

相容性条件：

① $\lim\limits_{x\to 0^+} u(x,0) = u(0,0) = \lim\limits_{t\to 0^+} u(0,t)$

$\quad \| \qquad\qquad\qquad\qquad \|$

$\quad \varphi(0) \qquad\qquad\qquad g(0)=0 \qquad\qquad \varphi(0)=0 \quad (4.25)$

② $\lim\limits_{x\to 0^+} u_t(x,0) = u_t(0,0) = \lim\limits_{t\to 0^+} u_t(0,t)$

$\quad \| \qquad\qquad\qquad\qquad \|$

$\quad \psi(0) \qquad\qquad\qquad g'(0)=0 \qquad\qquad \psi(0)=0 \quad (4.26)$

③ $\lim\limits_{x\to 0^+} u_{tt}(x,0) = u_{tt}(0,0) = \lim\limits_{t\to 0^+} u_{tt}(0,t)$

$\quad \| \qquad\qquad\qquad\qquad \|$

$\lim\limits_{x\to 0^+} (\partial_x^2 u(x,0) + f(x,0)) \qquad g''(0)=0 \qquad\qquad \varphi''(0)+f(0,0)=0 \quad (4.27)$

$\quad \|$

$\quad \varphi''(0)+f(0,0)$

thm 5

若 (4.21) 初值 $\varphi(x)\in C^2(R_+)$, $\psi(x)\in C^1(R_+)$, $f(x,t)\in C^1(\overline{R}_+\times\overline{R}_+)$

满足相容性条件，且边值 $g(t)\equiv 0$. (4.23). (4.24) 给出函数 $u\in C^2(\overline{R}_+\times\overline{R}_+)$

且为 (4.21) 的解

2. 若 $g(t)\neq 0$, 令 $v(x,t) = u(x,t) - g(t)$

则 $v(0,t) = u(0,t) - g(t) = 0$

原方程转化为

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$$\begin{cases} \partial_t^2 v - \partial_x^2 v = f(x,t) - g''(t) \\ v(x,0) = u(x,0) - g(0) = \varphi(x) - g(0) \\ \partial_t v(x,0) = \partial_t u(x,0) - g'(0) = \psi(x) - g'(0) \end{cases}$$

即转化为情形1.

再给出更一般相容性条件

thm 6 相容性条件

$\varphi(0) = g(0)$, $\psi(0) = g'(0)$, $f(0,0) + \varphi''(0) = g''(0)$

Rmk、相容性条件保证了初值与边值在二阶可微时的自洽.

thm 7 第二类边值问题

给定 $u_x(0,t) = g(t)$. 则令 $u(x,t) = x g(t) + v(x,t)$

$v(x,t)$ 满足 $v_x(0,t) = 0$. 之后利用偶延拓得到 $v(x,t)$, 进而得到 $u(x,t)$

§4.1.5 三维初值问题

$n=3$　$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \end{cases}$

在极坐标系下. $\Delta u = \partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u$　(Δ_{S^2} 表示 S^2 上 Laplace)

$\Delta \triangleq \operatorname{div} \nabla$

$$\partial_t^2 u - (\partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u) = f(x,t)$$

$$\int_{S^2} \Delta_{S^2} u\, d\omega = \int_{S^2} \operatorname{div} \nabla u\, d\omega = \int_{\partial S^2} \frac{\partial u}{\partial n} ds = 0$$

1, 先考虑 $f(x,t)=0$ 情形

$$\partial_t^2 u - (\partial_r^2 u + \frac{2}{r}\partial_r u + \frac{1}{r^2}\Delta_{S^2} u) = 0$$

对方程两边在 S^2 上积分，S^2 为单位球面

$$\partial_t^2 \int_{S^2} u\, d\omega - (\partial_r^2 \int_{S^2} u\, d\omega + \frac{2}{r}\partial_r \int_{S^2} u\, d\omega) = 0$$

令 $\bar{u}(t,r) = \frac{1}{4\pi}\int_{S^2} u\, d\omega$

则 $\partial_t^2 \bar{u} - \partial_r^2 \bar{u} - \frac{2}{r}\partial_r \bar{u} = 0$

令 $\bar{u}(t,r) = r^{-k} v(t,r)$

$$\partial_r \bar{u} = -k r^{-k-1} v(t,r) + r^{-k}\partial_r v$$

$$\partial_r^2 \bar{u} = -k[(-k-1) r^{-k-2} v(t,r) + r^{-k-1}\partial_r v] + (-k) r^{-k-1}\partial_r v + r^{-k}\partial_r^2 v$$

$$= k(k+1) r^{-k-2} v - 2k r^{-k-1}\partial_r v + r^{-k}\partial_r^2 v$$

$$\partial_r^2 \bar{u} + \frac{2}{r}\partial_r \bar{u} = k(k+1) r^{-k-2} v - 2k r^{-k-1}\partial_r v + r^{-k}\partial_r^2 v - 2k r^{-k-2} v + 2 r^{-k-1}\partial_r v$$

取 $k=1$, $v(t,r) = r\bar{u}(t,r)$

有 $\begin{cases} \partial_t^2 v - \partial_r^2 v = 0 \\ v(r,0) = r\bar{u}(r,0) = r\bar{\varphi}(r) \\ \partial_t v(r,0) = r\partial_t \bar{u}(r,0) = r\bar{\psi}(r) \end{cases}$　$r \geq 0$, "$-$" 表示积分

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将 v 关于 r 作偶延拓，得到 $\bar{v}$

以下由"‾"代表延拓后函数

$$\Rightarrow \bar{v}(r,t)=\frac{1}{2}\big((r+t)\bar{\varphi}(r+t)+(r-t)\bar{\varphi}(r-t)\big)+\frac{1}{2}\int_{r-t}^{r+t}y\,\bar{\Psi}(y)\,dy$$

Rmk.

$\partial_t^2 u-\Delta u=0$ 在以下变换中不变：

时间平移： $u(x,t)\mapsto u(x,t+t_0)$

空间平移： $u(x,t)\mapsto u(x+x_0,t)$

伸缩变换． $u(x,t)\mapsto u(\frac{x}{\lambda},\frac{t}{\lambda})\quad \lambda>0\quad \triangleq u^\lambda(x,t)$

$$\Big(\partial_t u^\lambda(x,t)=\partial_t\Big(u(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\Big)=(\partial_t u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\cdot\tfrac{1}{\lambda}$$

$$\partial_t^2 u^\lambda(x,t)=(\partial_t^2 u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\cdot\tfrac{1}{\lambda^2}$$

$$\text{故 }\partial_t^2 u^\lambda-\Delta u^\lambda=(\partial_t^2 u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})\cdot\tfrac{1}{\lambda^2}-\tfrac{1}{\lambda^2}(\Delta u)(\tfrac{x}{\lambda},\tfrac{t}{\lambda})=0\Big)$$

Lorentz 变换： $u(x,t)\to u\Big(x-x_v+\frac{x_v-vt}{\sqrt{1-|v|^2}},\ \frac{t-v\cdot x}{\sqrt{1-|v|^2}}\Big)$

$$x_v\triangleq\Big(x\cdot\frac{v}{|v|}\Big)\frac{v}{|v|}$$

以上方程中 $\bar{v}\to v\to\bar{u}$，下考察 $\bar{u}$ 与 u 的关系

$$\bar{u}(r,t)=\frac{1}{4\pi}\int_{S^2}u\,d\omega=\frac{1}{4\pi}\int_{S^2}u(r,\omega)\,d\omega$$

$$\bar{u}(0,t)=u(0,t)=\partial_r\big(r\bar{u}(r,t)\big)\Big|_{r=0}$$

$$=\partial_r V\Big|_{r=0}$$

$$=\frac{1}{2}\big(\bar{\varphi}(t)+t\bar{\varphi}'(t)+\bar{\varphi}(-t)-t\bar{\varphi}'(-t)\big)$$

$$+\frac{1}{2}\big(t\bar{\Psi}(t)-(-t)\bar{\Psi}(-t)\big)$$

$$=\bar{\varphi}(t)+t\bar{\varphi}'(t)+t\bar{\Psi}(t)$$

$$=\frac{d}{dt}\big(t\bar{\varphi}(t)\big)+t\bar{\Psi}(t)$$

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$$= \frac{d}{dt}\left(t\overline{\Phi}(t)\right) + t\Psi(t)$$

$$= \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(t\omega)\,d\omega\right) + \frac{t}{4\pi}\int_{S^2}\psi(t\omega)\,d\omega$$

对 $u(x+x_0, t)$ 应用于上一步,

其初值变为 $\varphi(x+x_0)$, $\psi(x+x_0)$

令 $x=0$,

$$u(x_0,t) = \frac{d}{dt}\left(\frac{t}{4\pi}\int_{S^2}\varphi(x_0+t\omega)\,d\omega\right) + \frac{t}{4\pi}\int_{S^2}\psi(x_0+t\omega)\,d\omega$$

$x_0 \to x$, $y = x + t\omega$

$$u(x,t) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$

(kirchhoff)

2. f 不恒为 0

$$u(x,t) = \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)\,dS(y)\right) + \frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$
$$+ \int_0^t \frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau} f(y,\tau)\,dy\,d\tau$$

Rmk 仅 n=3 时可良好转化为一维波方程问题

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§4.1.6 二维初值问题

$n=2$, $\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) \end{cases}$

1. $f(x,t)\equiv 0$

令 $\tilde{u}(\tilde{x},t)=u(\tilde{x},t)$ $\quad \tilde{x}=(x_1,x_2,x_3)$

$\tilde{\varphi}(\tilde{x})=\varphi(x_1,x_2)$

$\tilde{\psi}(\tilde{x})=\psi(x_1,x_2)$

则 $\tilde{u}(\tilde{x},t)$ 为三维波方程的解.

即 $\begin{cases} \partial_t^2\tilde{u}-\Delta\tilde{u}=0 \\ \tilde{u}(\tilde{x},0)=\tilde{\varphi}(\tilde{x}),\ \partial_t\tilde{u}(\tilde{x},0)=\tilde{\psi}(\tilde{x}) \end{cases}$

由 Kirchhoff,

$$\underset{\displaystyle u(x,t)}{\underset{\parallel}{\tilde{u}(\tilde{x},t)}}=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|\tilde{x}-\tilde{y}|=t}\tilde{\varphi}(\tilde{y})dS(\tilde{y})\right)+\frac{1}{4\pi t}\int_{|\tilde{x}-\tilde{y}|=t}\tilde{\psi}(\tilde{y})\,dS(\tilde{y})$$

$x_1,x_2,x_3=0$ 时, $\tilde{u}(0,t)=u(0,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y|=t}\varphi(y_1,y_2)\,dS(\tilde{y})\right)$

$$+\frac{1}{4\pi t}\int_{|y|=t}\psi(y_1,y_2)\,dS(\tilde{y})$$

注意到 $\int_{|y|=t}\psi(y_1,y_2)\,dS(\tilde{y})$

$$=2\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}}\psi(y_1,y_2)\,dS(\tilde{y})$$

$$\frac{\partial y_3}{y_1}=\frac{-y_1}{\sqrt{t^2-y_1^2-y_2^2}}$$

$$=2\int_{y_1^2+y_2^2\le t^2}\psi(y_1,y_2)\sqrt{1+\left(\frac{\partial y_3}{y_1}\right)^2+\left(\frac{\partial y_3}{y_2}\right)^2}\,dy_1dy_2$$

$$=2\int_{y_1^2+y_2^2\le t^2}\psi(y_1,y_2)\frac{t}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2$$

$$\Rightarrow u(0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{B(0,t)}\frac{\varphi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}dy_1dy_2\right)+\frac{1}{2\pi}\int_{B(0,t)}\frac{\psi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}dy_1dy_2$$

对 $u(x+x_0,t)$ 利用结论

$$u(x_0,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{|y|<t}\frac{\varphi(y+x_0)}{\sqrt{t^2-y^2}}dy\right)+\frac{1}{2\pi}\int_{|y|<t}\frac{\psi(y+x_0)}{\sqrt{t^2-y^2}}dy$$

由 x_0 任意性 $u(x,t)=\frac{d}{dt}\left(\frac{1}{2\pi}\int_{|y-x|<t}\frac{\varphi(y)}{\sqrt{t^2-(y-x)^2}}dy\right)+\frac{1}{2\pi}\int_{|y-x|<t}\frac{\psi(y)}{\sqrt{t^2-(y-x)^2}}dy$

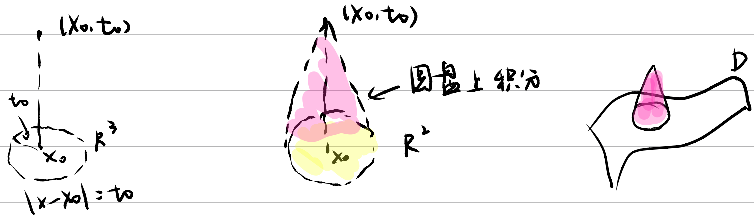
2、$f\neq 0$

3、性质

$$u(x,t)=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|y-x|=t}\varphi(y)dS(y)\right)+\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)dS(y)$$

(kirchhoff)

(x_0,t_0) 仅与初值在球面上积分有关



依赖区域：(x_0,t_0) 依赖于 $\underset{\substack{\parallel\triangle \\ D_{x_0,t_0}}}{\{x \mid |x-x_0|\le t_0\}}$ 的值

决定区域：$\{(x,t) \mid D_{x,t}\subset D\}$ 为 D 的决定区域

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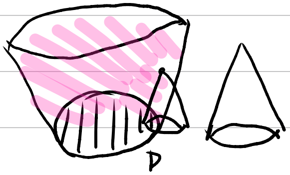


影响区域

Rmk. 几何性质

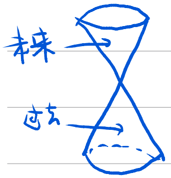
1、波方程具有有限传播速度

2、Huygens原理

n=3, 依赖于球面

n=2, 依赖于圆盘内

3、



光锥 因果律

§4.1.7 能量估计

$u_{tt}-\Delta u=0 \Rightarrow u_t(u_{tt}-\Delta u)=0$

$u_t u_{tt}=\frac{1}{2}\partial_t(u_t)^2$

$$
\begin{aligned}
u_t\Delta u &= \sum_{i=1}^{n} u_t u_{x_i^2} = \sum_{i=1}^{n}\left[\partial x_i(u_t u_{x_i})-\partial_t\partial x_i u\, u_{x_i}\right] \\
&= \sum_{i=1}^{n}\left[\partial x_i(u_t u_{x_i})-\frac{1}{2}\partial_t(u_{x_i})^2\right] \\
&= \operatorname{div}(u_t\nabla u)-\frac{1}{2}\partial_t|\nabla u|^2
\end{aligned}
$$

$\Rightarrow \partial_t\left[\frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2\right]-\operatorname{div}(u_t\nabla u)=0$

称作能量守恒微分形式

u、偏导数在空间无穷远 $\to 0$

则 $\partial_t\int_{R^n}\left(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2\right)dx-\int_{R^n}\operatorname{div}(u_t\nabla u)=0$

散度 thm. $\int_\Omega \operatorname{div}\vec{F}\,dx=\int_{\partial\Omega}\vec{F}\cdot\vec{n}\,ds$

$\Rightarrow \partial_t\int_{R^n}\left(\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2\right)dx=0$

令 $E(t)=\int_{R^n}\underbrace{\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2}_{\text{能量密度 } e}dx$，则 $E(t)=E(0)$

称作能量守恒积分形式

考虑 $\begin{cases}\partial_t^2u-\Delta u=0 & x\in\Omega\subset R^n, t>0 \\ u|_{\partial\Omega}=0, & u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x)\end{cases}$

$\partial_t\left[\frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2\right]=\operatorname{div}(u_t\nabla u)$

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{1}{2}(u_t)^2+\frac{1}{2}|\nabla u|^2dx &= \int_\Omega \operatorname{div}(u_t\nabla u)\,dx \\
&= \int_{\partial\Omega} u_t\nabla u\cdot\vec{n}\,ds \\
&= \int_{\partial\Omega} u_t\frac{\partial u}{\partial n}\,ds
\end{aligned}
$$

$u|_{\partial\Omega}=0 \Rightarrow u_t|_{\partial\Omega}=0 \Rightarrow E(t)=\int_\Omega\frac{1}{2}u_t^2+\frac{1}{2}|\nabla u|^2$，有 $E(t)=E(0)$

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考虑 $\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & x\in\Omega \quad \text{初值} \quad (※) \\ u|_{\partial\Omega} = h(x,t) & t\geq 0 \quad \text{第一边值} \end{cases}$

该方程最多有一个古典解.

pr: 设 (※) 有两个解 u_1, u_2.

令 $u(x,t) = u_1(x,t) - u_2(x,t)$

则 $\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0)=0,\ u_t(x,0)=0 \\ u|_{\partial\Omega}=0 \end{cases}$

由能量估计, 令 $E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 dx$

则 $E(t) = E(0) = 0$

$\Rightarrow u_t = 0,\ \nabla u = \vec{0} \quad \forall x \text{ in } \Omega,\ t\geq 0$

$\Rightarrow u = \text{const} \quad \text{in } \Omega$

由于 u 边值为 0, 为使得 u 连续至边界, $u\equiv 0$

Rmk. 对第二类边值, 能量估计时

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 dx = \int_{\partial\Omega} u_t \frac{\partial u}{\partial \vec{n}} ds$$

给出 $\frac{\partial u}{\partial \vec{n}}=0$ 的条件可类似 u_t 处理

对第三类边值, $\frac{\partial u}{\partial \vec{n}} + \alpha u = 0\ (\alpha>0),\ x\in\partial\Omega$

def. $E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} \alpha u^2 dx$

此时 $\frac{dE(t)}{dt} = 0$

考虑
$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & x\in\Omega,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & x\in\Omega \quad \text{初值} \quad (※※) \\ u|_{\partial\Omega}=0 & t\geq 0 \quad \text{第一边值} \end{cases}$$

(※※)的解在下述意义下关于初值和右端项稳定:

$\forall \varepsilon>0,\ \exists \eta=\eta(\varepsilon,T)$, s.t.

若 $\|\varphi_1-\varphi_2\|_{L^2(\Omega)}\leq\eta$, $\|\nabla\varphi_1-\nabla\varphi_2\|_{L^2(\Omega)}\leq\eta$, $\|\psi_1-\psi_2\|_{L^2(\Omega)}\leq\eta$

$\|f_1-f_2\|_{L^2(0,T,\Omega)}\leq\eta$

则以 φ_1,ψ_1 为初值, f_1 为右端项的解 u_1

与以 φ_2,ψ_2 为初值, f_2 为右端项的解 u_2

其差在 $0\leq t\leq T$ 上满足 $\|u_1-u_2\|_{L^2(\Omega)}+\|\nabla u_1-\nabla u_2\|_{L^2(\Omega)}\leq\varepsilon$

$$\|\partial_t u_1-\partial_t u_2\|_{L^2(\Omega)}\leq\varepsilon$$

Rmk. $\|f\|_{L^2(\Omega)}=\left(\int_\Omega |f(x)|^2dx\right)^{\frac12}$

$\|f\|_{L^2(0,T,\Omega)}=\left(\int_0^T\int_\Omega |f(x,t)|^2dxdt\right)^{\frac12}$

pr: 令 $u(x,t)=u_1(x,t)-u_2(x,t)$

$f=f_1-f_2,\ \varphi=\varphi_1-\varphi_2,\ \psi=\psi_1-\psi_2$

则
$$\begin{cases} \partial_t^2 u-\Delta u=f_1-f_2=f \\ u(x,0)=\varphi_1-\varphi_2=\varphi,\ u_t(x,0)=\psi_1-\psi_2=\psi \\ u|_{\partial\Omega}=0 \end{cases}$$

$$u_t(\partial_t^2 u-\Delta u)=\partial_t\left[\tfrac12(u_t)^2+\tfrac12|\nabla u|^2\right]-\mathrm{div}(u_t\nabla u)=u_t\cdot f$$

在 Ω 上积分, $\partial_t\int_\Omega \frac12(u_t)^2+\frac12|\nabla u|^2dx=\int_\Omega u_t f\,dx\leq\int_\Omega \frac12 f^2+\frac12(u_t)^2dx$

$$\frac{d}{dt}E(t)\leq\frac12\int_\Omega f^2dx+\frac12\int_\Omega(u_t)^2dx\leq\int_\Omega\frac12 f^2dx+E(t)$$

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$\frac{d}{dt}(e^{-t}E(t)) \le \frac{1}{2}e^{-t}\int_\Omega f^2 dx$

由Gronwall, $e^{-t}E(t) - E(0) \le \frac{1}{2}\int_0^t e^{-s}\int_\Omega f^2(x,s)dxds$

$\Rightarrow E(t) \le e^t(E(0) + \frac{1}{2}\int_0^t e^{-s}\int_\Omega f^2(x,s)dxds)$

$\le e^t(E(0) + \frac{1}{2}\int_0^t\int_\Omega f^2(x,s)dxds)$

$\le e^T(E(0) + \frac{1}{2}\int_0^T\int_\Omega f^2(x,s)dxds)$

$\le C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$ $\forall\, 0\le t\le T$ C_{1T} = const. 与T有关

$E(0)$表示 $\int_\Omega \frac{1}{2}(u_t)^2 + \frac{1}{2}|\nabla u|^2 dx$ 的初值

令 $y(t) = \int_\Omega |u|^2 dx$

与T有关常数

$y'(t) = 2\int_\Omega u u_t dx \le \int_\Omega u^2 + u_t^2 dx \le y(t) + 2E(t)$

$\le y(t) + 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$

$\frac{d}{dt}(e^{-t}y(t)) \le 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$

由Gronwall, $e^{-t}y(t) - y(0) \le t\cdot 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$

$\Rightarrow y(t) \le e^t[y(0) + t\cdot 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)]$

$\le e^t[y(0) + T\cdot 2C_{1T}(E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)]$

$\le C_{2T}(y(0) + E(0) + \int_0^T\int_\Omega f^2(x,s)dxds)$

故 $\|u(t)\|^2_{L^2(\Omega)} + \|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}$

$\le C_T(\|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega)} + \int_0^T\int_\Omega f^2(x,s)dxds)$

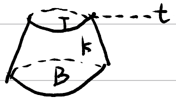
∃与T有关的常数

$\le 4\eta^2 C_T \triangleq \frac{\varepsilon}{2} < \varepsilon$

考虑 $\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \end{cases} \quad x \in \mathbb{R}^3$

两边同乘 u_t，有 $\partial_t e(u) - \operatorname{div}(u_t \nabla u) = 0$

在锥台 $|x - x_0| \leq R - t$ 上积分



$$\iint_{\triangle} \partial_t e(u) - \operatorname{div}(u_t \nabla u)\, dx\, dt = 0$$

$$左 = \iint_{\triangle} (\partial_t, \nabla_x) \cdot (e(u), -u_t \nabla u)\, dx\, dt$$

$$= \int_{\partial \triangle} (e(u), -u_t \nabla u) \cdot \vec{n}\, ds$$

$$= -\int_B e(u)(0)\, dx + \int_T e(u)(t_0)\, dx$$

表示时间取值

$K:\ |x - x_0|^2 = |R - t|^2$

$\varphi(x,t) = |x - x_0|^2 - |R - t|^2 = 0$

$\nabla \varphi = (2(R-t),\ 2(x - x_0))$

$n = \frac{\pm \nabla \varphi}{\|\nabla \varphi\|} = \pm \frac{1}{\sqrt{2}} \left(\frac{R-t}{|R-t|},\ \frac{x - x_0}{|x - x_0|} \right)$

$$+ \int_K (e(u), -u_t \nabla u) \cdot \frac{1}{\sqrt{2}} \left(\frac{R-t}{|R-t|},\ \frac{x - x_0}{|x - x_0|} \right)$$

$$\Rightarrow \int_B e(u)(0)\, dx = \int_T e(u)(t_0)\, dx + \frac{1}{\sqrt{2}} \int_K \left(\frac{1}{2}(u_t)^2 + \frac{1}{2} |\nabla u|^2 \right) \frac{R-t}{|R-t|} - u_t \nabla u \cdot \frac{x - x_0}{|x - x_0|}\, ds$$

$$= \int_T e(u)(t_0)\, dx + \frac{1}{2\sqrt{2}} \int_K (u_t)^2 + |\nabla u|^2 - 2 u_t \nabla u \cdot \frac{x - x_0}{|x - x_0|}\, ds$$

$$= \int_T e(u)(t_0)\, dx + \underbrace{\frac{1}{2\sqrt{2}} \int_K \left| u_t - \frac{x - x_0}{|x - x_0|} \cdot \nabla u \right|^2 + |\nabla u|^2 - \left| \frac{x - x_0}{|x - x_0|} \cdot \nabla u \right|^2 ds}_{\text{Flux } [0,t] \geq 0}$$

即 $\int_B e(u)(0)\, dx = \int_T e(u)(t_0)\, dx + \text{Flux } [0,t]$

（$t=0$ 处能量）　（$t=t_0$ 处能量）　（能量溢出）

若 $(u, u_t)\big|_{t=0} = 0$，在 B 上能量为 0，则在 (u, u_t) 在 T 上恒为 0

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§4.2 混合问题

混合问题即初边值问题

$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x) \qquad x\in\Omega,\ t>0 \\ u\big|_{\partial\Omega} = g(x,t) \,/\, \frac{\partial u}{\partial n}\big|_{\partial\Omega} = g(x,t) \,/\, \frac{\partial u}{\partial n} + \alpha u\big|_{\partial\Omega} = g(x,t) \end{cases}$$

4.2.1 常微分方程齐次边值问题

考虑
$$\begin{cases} X''(x) + \lambda X(x) = 0 \quad x\in(0,l) \\ -\alpha_1 X'(0) + \beta_1 X(0) = 0 \qquad (3.15) \\ \alpha_2 X'(l) + \beta_2 X(l) = 0 \end{cases}$$

$\alpha_i \geq 0,\ \beta_i \geq 0,\ \alpha_i + \beta_i > 0,\ i = 1,2$

称为 Sturm-Liouville 特征值问题，λ称为特征值

$\lambda \in R$ 称为特征值，相应于λ的非零解 $X(x)$ 称为对应于这个特征值的特征函数

thm (1) 所有特征值为非负实数

$\beta_1 + \beta_2 > 0$ 时，所有特征值为正数

(2) 不同特征值对应特征函数正交.

即 $\int_0^l X_\lambda(x) X_\mu(x)\, dx = 0$

(3) $\lambda_1, \ldots, \lambda_n \ldots$ 为特征值

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \qquad \lim_{n\to\infty} \lambda_n = +\infty$

(4) $f(x) \in L^2(0,l)$ 可按特征函数系展开为

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x)$$

$$C_n = \frac{\int_0^l f(x) X_n(x)\, dx}{\int_0^l X_n^2(x)\, dx}$$

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pr:(1) $X''+\lambda X=0 \Rightarrow XX''+\lambda X^2=0$

$XX''+\lambda X^2=(XX')'-(X')^2+\lambda X^2=0$

$$\Rightarrow \lambda\int_0^l X^2(x)dx=\int_0^l (X'(x))^2dx-\int_0^l (X(x)X'(x))'dx$$
$$=\int_0^l (X'(x))^2dx-X(x)X'(x)\Big|_{x=0}^{x=l}$$
$$=\int_0^l (X'(x))^2dx+X(0)X'(0)-X(l)X'(l)$$

$-\alpha_1 X'(0)+\beta_1 X(0)=0 \Rightarrow -\alpha_1 X'(0)^2+\beta_1 X(0)X'(0)=0$

$-\alpha_1 X'(0)X(0)+\beta_1 X(0)^2=0$

$\Rightarrow X(0)X'(0)=\dfrac{\alpha_1}{\alpha_1+\beta_1}X'(0)^2+\dfrac{\beta_1}{\alpha_1+\beta_1}X(0)^2$

类似地 $X(l)X'(l)=-\dfrac{\alpha_2 X'(l)^2+\beta_2 X(l)^2}{\alpha_2+\beta_2}$

代入表达式有

$$\lambda\int_0^l X^2dx=\int_0^l (X'(x))^2dx+\frac{\alpha_1}{\alpha_1+\beta_1}X'(0)^2+\frac{\beta_1}{\alpha_1+\beta_1}X(0)^2+\frac{\alpha_2}{\alpha_2+\beta_2}X'(l)^2+\frac{\beta_2}{\alpha_2+\beta_2}X(l)^2$$

$\geq 0 \Rightarrow \lambda\geq 0$

$\lambda=0 \Leftrightarrow X'=0$，且 $\dfrac{\beta_1}{\alpha_1+\beta_1}X(0)^2+\dfrac{\beta_2}{\alpha_2+\beta_2}X(l)^2=0$

故 $\beta_1=\beta_2=0$ 时，$X(x)=\mathrm{Const}$

若 β_1、β_2 中存在不为0，$X(x)=\mathrm{Const}=0$，0不为特征值

(2) 设 X_λ、X_μ 为不同特征值 λ、μ 的特征函数

则 $X_\lambda''+\lambda X_\lambda=0$

$X_\mu''+\mu X_\mu=0$

$$\lambda\int_0^l X_\lambda X_\mu dx=-\int_0^l X_\mu X_\lambda''dx=-\int_0^l X_\mu d(X_\lambda')=\int_0^l X_\lambda' X_\mu' dx-X_\mu X_\lambda'\Big|_0^l$$
$$=X_\mu(0)X_\lambda'(0)-X_\mu(l)X_\lambda'(l)+\int_0^l X_\lambda' X_\mu' dx$$

同理有结果，$\mu\int_0^l X_\lambda X_\mu dx=X_\lambda(0)X_\mu'(0)-X_\mu'(l)X_\lambda(l)+\int_0^l X_\lambda' X_\mu' dx$

相减可得

$$(\lambda-\mu)\int_0^l X_\lambda X_\mu dx=(X_\lambda'(0)X_\mu(0)-X_\mu(0)X_\mu'(0))-(X_\mu(l)X_\lambda'(l)-X_\mu'(l)X_\lambda(l))$$

由边条件 $\begin{cases}-\alpha_1 X_\lambda'(0)+\beta_1 X_\lambda(0)=0 & (1)\\ \alpha_2 X_\lambda'(l)+\beta_2 X_\lambda(l)=0 & (2)\end{cases}$

$\begin{cases}-\alpha_1 X_\mu'(0)+\beta_1 X_\mu(0)=0 & (3)\\ \alpha_2 X_\mu'(l)+\beta_2 X_\mu(l)=0 & (4)\end{cases}$

(1)(3) 构成关于 α_1、β_1 的线性方程，有非零解，则

$$\begin{vmatrix}X_\lambda'(0) & X_\lambda(0)\\ X_\mu'(0) & X_\mu(0)\end{vmatrix}=0 \Rightarrow X_\lambda'(0)X_\mu(0)-X_\lambda(0)X_\mu'(0)=0$$

同理 $X_\mu(l)X_\lambda'(l)-X_\mu'(l)X_\lambda(l)=0$

$\lambda\neq\mu \Rightarrow \int_0^l X_\lambda X_\mu dx=0$

Rmk. (3)(4) 的证明在泛函分析中学到

构成 L^2 一组基是由于算子 $\frac{d^2}{dt^2}$ 为对称紧算子

§4.2.2 分离变量法

考虑
$$\begin{cases} \partial_t^2 u-\partial_x^2 u=f(x,t) & 0<x<l,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & 0\le x\le l \\ u(0,t)=g_1(t),\ u(l,t)=g_2(t) & t\geq 0 \end{cases}$$

(一维定长弦的振动)

1、$f\equiv 0$，$g_1(t)$、$g_2(t)\equiv 0$

则
$$\begin{cases} \partial_t^2 u-\partial_x^2 u=0 & 0<x<l,\ t>0 \\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x) & 0\le x\le l \\ u(0,t)=0,\ u(l,t)=0 & t\geq 0 \end{cases}$$

① u 仅与 t 有关，则 $\partial_t^2 u=0 \Rightarrow u=C_1t+C_2$

$u(0,t)=C_1t+C_2$ 不满足边值

② u 仅与 x 有关，则 $\partial_t^2 x=0 \Rightarrow u=C_1x+C_2$

$u(x,0)=C_1x+C_2=\varphi(x)$ 不容易满足初值

③ 令 $u(x,t)=T(t)X(x)$

则 $T''(t)X(x)-T(t)X''(x)=0$

$T(t)\cdot X(x)\neq 0 \Rightarrow \dfrac{T''(t)}{T(t)}=\dfrac{X''(x)}{X(x)}\triangleq -\lambda$

$$\Rightarrow \begin{cases} X''(x)+\lambda X(x)=0 \\ T''(t)+\lambda T(t)=0 \end{cases}$$

$$\begin{cases} u(0,t)=0 \\ u(l,t)=0 \end{cases} \Rightarrow \begin{cases} T(t)X(0)=0 \\ T(t)X(l)=0 \end{cases} \quad \forall t\geq 0$$

$\Rightarrow X(0)=X(l)=0$

考虑对于X的Sturm-Liouville 边值问题

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0 \\ X(l)=0 \end{cases}$$

$$0=\int_0^l X(x)\left(X''(x)+\lambda X(x)\right)dx = X(x)X'(x)\Big|_0^l - \int_0^l (X'(x))^2dx + \lambda\int_0^l (X(x))^2dx$$

$$\Rightarrow \lambda\int_0^l (X(x))^2dx = \int_0^l (X'(x))^2dx$$

$$\Rightarrow \lambda \geq 0$$

若 $\lambda=0$，则 $X''(x)=0 \Rightarrow X(x)=C_1x+C_2$

$X(0)=C_2=0$，$X(l)=C_1 l=0 \Rightarrow C_1=0 \Rightarrow X(x)\equiv 0$

若 $\lambda>0$，$X''(x)+\lambda X(x)=0$

$\Rightarrow X(x)=C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

$X(0)=C_1=0$，$X(l)=C_2\sin\sqrt{\lambda}l=0$ 　$\sqrt{\lambda}l=n\pi$ $n\in\mathbb{Z}^*$

$\Rightarrow \lambda_n=\left(\frac{n\pi}{l}\right)^2$ 　$n=1.2\cdots$

与之对应特征函数 $X_n(x)=\sin(\frac{n\pi}{l}x)$ (事实上为 $C_2\sin(\frac{n\pi}{l}x)$，但系数可被"C_n""D_n"吸收)

由于 $T_n''(t)+\lambda_n T_n(t)=0$

$T_n(t)=C_n\cos(\frac{n\pi}{l}t)+D_n\sin(\frac{n\pi}{l}t)$

令 $u(x,t)=\sum_{n=1}^{\infty}\left[C_n\cos(\frac{n\pi}{l}t)+D_n\sin(\frac{n\pi}{l}t)\right]\sin(\frac{n\pi}{l}x)$

再利用初值条件，

$u(x,0)=\sum_{n=1}^{\infty} C_n\sin(\frac{n\pi}{l}x)=\varphi(x)$

$u_t(x,0)=\sum_{n=1}^{\infty}\frac{n\pi}{l}D_n\sin(\frac{n\pi}{l}x)=\psi(x)$

由 Sturm-Liouville，$\left\{\sin(\frac{n\pi}{l}x)\right\}_{n=1}^{\infty}$ 为 L^2 中一组正交基

$$\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin(\frac{n\pi}{l}x)$$

$$\varphi_n=\frac{\int_0^l\varphi(x)\sin\frac{n\pi}{l}x\,dx}{\int_0^l\sin^2(\frac{n\pi}{l}x)\,dx}=\frac{2}{l}\int_0^l\varphi(x)\sin(\frac{n\pi}{l}x)\,dx$$

$$\psi(x)=\sum_{n=1}^{\infty}\psi_n\sin(\frac{n\pi}{l}x)$$

$$\psi_n=\frac{\int_0^l\psi(x)\sin(\frac{n\pi}{l}x)\,dx}{\int_0^l\sin^2(\frac{n\pi}{l}x)\,dx}=\frac{2}{l}\int_0^l\psi(x)\sin(\frac{n\pi}{l}x)\,dx$$

$$\Rightarrow\left.\begin{aligned}C_n&=\varphi_n=\frac{2}{l}\int_0^l\varphi(x)\sin(\frac{n\pi}{l}x)\,dx\\D_n&=\frac{l}{n\pi}\psi_n=\frac{2}{n\pi}\int_0^l\psi(x)\sin(\frac{n\pi}{l}x)\,dx\end{aligned}\right\}(※)$$

故原方程解为

$$u(x,t)=\sum_{n=1}^{\infty}\left[C_n\cos(\frac{n\pi}{l}t)+D_n\sin(\frac{n\pi}{l}t)\right]\sin(\frac{n\pi}{l}x)$$

其中 C_n、D_n 由 (※) 给出

thm. 相容性条件

若 $\varphi\in C^3([0,l])$, $\psi\in C^2([0,l])$, $\varphi(x)$、$\psi(x)$ 在 $(0,l)\times(0,+\infty)\triangleq Q$

角点满足相容性条件 $\varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=\psi(0)=\psi(l)=0$

则 $u(x,t)\in C^2(\bar{Q})$ 为古典解

Rmk. 若 $u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$

T_i、X_i 均足够好，使微数求导与求和可换序

则 $\partial_t^2u-\partial_x^2u=\sum_{n=1}^{\infty}T_n''(t)X_n(x)-T_n(t)X_n''(x)$

$$=\sum_{n=1}^{\infty}\left(T_n''(t)+\lambda_nT_n(t)\right)X_n(x)=0$$

2. $f\not\equiv0$, $g_1(t)$、$g_2(t)\equiv0$

令 $f(x,t)=\sum_{n=1}^{\infty}f_n(t)\sin(\frac{n\pi}{l}x)$

$$\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin(\frac{n\pi}{l}x),\quad\psi(x)=\sum_{n=1}^{\infty}\psi_n\sin(\frac{n\pi}{l}x)$$

令 $u(x,t)=\sum_{n=1}^{\infty} T_n(t)\sin(\frac{n\pi}{l}x)$

代入 $\partial_t^2 u-\partial_x^2 u=f(x,t)$

有 $\sum_{n=1}^{\infty}(T_n''(t)+\lambda_n T_n(t))\sin(\frac{n\pi}{l}x)=\sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi}{l}x)$

则 T_n 满足方程 $T_n''(t)+\lambda_n T_n(t)=f_n(t)$

$u(x,0)=\sum_{n=1}^{\infty} T_n(0)\sin(\frac{n\pi}{l}x)=\sum_{n=1}^{\infty}\varphi_n\sin(\frac{n\pi}{l}x) \Rightarrow T_n(0)=\varphi_n$

$u_t(x,0)=\sum_{n=1}^{\infty} T_n'(0)\sin(\frac{n\pi}{l}x)=\sum_{n=1}^{\infty}\psi_n\sin(\frac{n\pi}{l}x) \Rightarrow T_n'(0)=\psi_n$

$\Rightarrow T_n(t)=\varphi_n\cos(\frac{n\pi}{l}t)+\frac{l}{n\pi}\psi_n\sin(\frac{n\pi}{l}t)+\frac{l}{n\pi}\int_0^t f_n(\tau)\sin(\frac{n\pi}{l}(t-\tau))\,d\tau \quad (※)$

故原方程解为

$u(x,t)=\sum_{n=1}^{\infty} T_n(t)\sin(\frac{n\pi}{l}x)$ T_n 由 (※) 给出

3. $f\not\equiv 0$, $g_1, g_2\not\equiv 0$

令 $v(x,t)=u(x,t)-\frac{(l-x)g_1(t)+xg_2(t)}{l}$

则 $\left.\begin{array}{l} v(0,t)=u(0,t)-g_1(t)=0 \\ v(l,t)=u(l,t)-g_2(t)=0 \end{array}\right\}$ 边值化为0

$$\begin{cases} \partial_t^2 v-\partial_x^2 v=f(x,t)-\frac{(l-x)g_1''(t)+xg_2''(t)}{l} \\ v(x,0)=\varphi(x)-\frac{(l-x)g_1(0)+xg_2(0)}{l} \\ v_t(x,0)=\psi(x)-\frac{(l-x)g_1'(0)+xg_2'(0)}{l} \end{cases}$$

进而转化为零边值问题

Rmk. 本质为 S-L 边值问题的特征函数展开，与方程类型无关.

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ex. (热传导方程)

$$\begin{cases} u_t = u_{xx} & 0<x<l \\ u(x,0) = \varphi(x) \\ u(0,t) = 0,\ u_x(l,t) + h u(l,t) = 0 & h>0 \end{cases}$$

设 $u(x,t) = T(t)X(x)$

则 $T'(t)X(x) = T(t)X''(x)$

$$\Rightarrow \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} \triangleq -\lambda$$

$u(0,t) = 0 \Rightarrow T(t)X(0) = 0$

$u_x(l,t) + h u(l,t) = 0 \Rightarrow T(t)X'(l) + h T(t) X(l) = 0$

对 $\forall t$ 成立. 只需 $X'(l) + hX(l) = 0$

故 $X(x)$ 满足边值问题

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X'(l) + hX(l) = 0 \end{cases}$$

① $\lambda<0$ 时, $X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

$X(0) = 0 \Rightarrow C_1 + C_2 = 0$

$$C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}l} - C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} + h(C_1e^{\sqrt{-\lambda}l} + C_2e^{-\sqrt{-\lambda}l}) = 0$$

$$C_1\left[(\sqrt{-\lambda}e^{\sqrt{-\lambda}l} + \sqrt{-\lambda}e^{-\sqrt{-\lambda}l}) + h(e^{\sqrt{-\lambda}l} - e^{-\sqrt{-\lambda}l})\right] = 0$$

$$\Leftrightarrow C_1 = 0 \text{ 或 } \begin{cases} \sqrt{-\lambda} + h = 0 \\ \sqrt{-\lambda} - h = 0 \end{cases} \Rightarrow h = 0$$

故 $C_1 = 0,\ C_2 = 0$

Rmk. 利用本方法同样可说明 $\lambda \geq 0$, 同乘 x 为 S-L 问题的手段

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② $\lambda=0$ 时, $X(x)=C_1x+C_2$

$X(0)=C_2=0$

$X'(l)+hX(l)=C_1+hC_1l=0 \Rightarrow C_1=0 \Rightarrow X(x)\equiv 0$

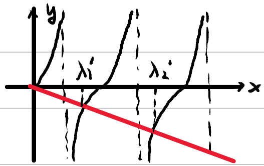
故 $\lambda>0$, 则 $X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

$X(0)=C_1=0$

$X'(l)+hX(l)=C_2\sqrt{\lambda}\cos(\sqrt{\lambda}l)+hC_2\sin(\sqrt{\lambda}l)=0$

$C_2\neq 0 \Rightarrow \tan(\sqrt{\lambda}l)=\frac{-\sqrt{\lambda}}{h}$

考察方程 $\tan x=\frac{-x}{hl}$ 的解



$\exists\ 0<\lambda_1<\cdots<\lambda_n<\cdots$, 满足 $\tan(\sqrt{\lambda}l)=\frac{-\sqrt{\lambda}}{h}$

令 $X_n(x)=\sin(\sqrt{\lambda_n}x)$ 为 λ_n 对应的特征函数

$T_n(t)$ 满足方程 $T_n'+\lambda_nT_n=0 \Rightarrow T_n(t)=A_ne^{-\lambda_nt}$

则 $u(x,t)=\sum_{n=1}^{\infty}A_ne^{-\lambda_nt}\sin(\sqrt{\lambda_n}x)$

而 $u(x,0)=\sum_{n=1}^{\infty}A_n\sin(\sqrt{\lambda_n}x)=\varphi(x)=\sum_{n=1}^{\infty}\varphi_n\sin(\sqrt{\lambda_n}x)$

$$\Rightarrow A_n=\varphi_n=\frac{\int_0^l\varphi(x)\sin(\sqrt{\lambda_n}x)dx}{\int_0^l\sin^2(\sqrt{\lambda_n}x)dx}$$

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ex. (位势方程)

令 $\Omega=\{(x,y) \mid x^2+y^2<1\}$

考虑Ω上位势方程:

$$\begin{cases} \Delta u=0 & \text{in } \Omega \\ u=\varphi & \text{in } \partial\Omega \end{cases}$$

令 $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

$$\Rightarrow \begin{cases} \partial_r^2 u+\frac{1}{r}\partial_r u+\frac{1}{r^2}\partial_\theta^2 u=0 & \text{in } \Omega \\ u\big|_{r=1}=\varphi(\cos\theta,\sin\theta)\triangleq\tilde{\varphi}(\theta) \end{cases}$$

令 $u(r,\theta)=R(r)\Theta(\theta)$

则 $R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$

$$\Rightarrow -r^2\frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}=\frac{\Theta''(\theta)}{\Theta(\theta)}\triangleq-\lambda$$

考察$\Theta(\theta)$满足的方程

$$\begin{cases} \Theta''(\theta)+\lambda\Theta(\theta)=0 \\ \Theta(\theta)=\Theta(\theta+2\pi) \end{cases}$$

若$\lambda<0$, $\Theta(\theta)=C_1e^{-\sqrt{\lambda}\theta}+C_2e^{\sqrt{\lambda}\theta}$, 不以$2\pi$为周期

若$\lambda=0$, $\Theta(\theta)=C_1\theta+C_2$, 不以$2\pi$为周期, 除非$C_1=0$, 即$\Theta(\theta)$为常数

若$\lambda>0$, $\Theta(\theta)=C_1\cos(\sqrt{\lambda}\theta)+C_2\sin(\sqrt{\lambda}\theta)$

以2π为周期 $\Leftrightarrow \sqrt{\lambda}\in\mathbb{Z}^*$

则 $\Theta_n(\theta)=C_n\cos(n\theta)+D_n\sin(n\theta)$ 为 $\lambda_n=n^2$ 对应特征函数, $n=1,2\cdots$

可构成空间一组基 由于 $\frac{d^2}{dt^2}$ 为对称紧算子

(补充$\Theta_0(\theta)=C_0$)

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考察 $R(r)$ 满足的方程

$$r^2R_n''(r)+rR_n'(r)-n^2R_n=0$$

为 Euler 方程.

令 $r=e^t$, $\frac{dR_n}{dt}=R_n'(r)r$, $\frac{d^2R_n}{dt^2}=R_n''(r)r^2+R_n'(r)r=R_n''(r)r^2+\frac{dR_n}{dt}$

故 $\frac{d^2R_n}{dt^2}-n^2R_n=0$

$$\Rightarrow R_n(r)=\begin{cases} C_1e^{nt}+C_2e^{-nt}=C_1r^n+C_2r^{-n} & n\neq 0 \\ C_1t+C_2=C_1\ln r+C_2 & n=0 \end{cases}$$

为使 $r=0$ 处连续可微, 应满足 $R_n(r)=\begin{cases} C_1r^n & n\neq 0 \\ C_2 & n=0 \end{cases}$

取 $C_1=C_2=1$.

设 $u(r,\theta)=C_0+\sum_{n=1}^{\infty}r^n(C_n\cos(n\theta)+D_n\sin(n\theta))$

$u|_{r=1}=C_0+\sum_{n=1}^{\infty}C_n\cos(n\theta)+D_n\sin(n\theta)=\tilde{\varphi}(\theta)$

$C_n\int_0^{2\pi}\cos^2(n\theta)\,d\theta=\int_0^{2\pi}\tilde{\varphi}(\theta)\cos(n\theta)\,d\theta \Rightarrow C_n=\frac{1}{\pi}\int_0^{2\pi}\tilde{\varphi}(\theta)\cos(n\theta)\,d\theta$

$D_n\int_0^{2\pi}\sin^2(n\theta)\,d\theta=\int_0^{2\pi}\tilde{\varphi}(\theta)\sin(n\theta)\,d\theta \Rightarrow D_n=\frac{1}{\pi}\int_0^{2\pi}\tilde{\varphi}(\theta)\sin(n\theta)\,d\theta$

$C_0\int_0^{2\pi}d\theta=\int_0^{2\pi}\tilde{\varphi}(\theta)\,d\theta \Rightarrow C_0=\frac{1}{2\pi}\int_0^{2\pi}\tilde{\varphi}(\theta)\,d\theta$

Rmk. 分离变量法对区域、算子较为敏感