Week 5

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2024 年 4 月 11 日

9.5.12 曲面 $z = xe^{x/y}$ 上所有切平面都通过原点。

解.

$$F = xe^{x/y} - z, \boldsymbol{J}F = (e^{x/y} + \frac{x}{y}e^{x/y}, -\frac{x^2}{y^2}e^{x/y}, -1),$$

在 (x_0, y_0, z_0) 处的切平面方程为

$$\left(e^{x_0/y_0} + \frac{x_0}{y_0}e^{x_0/y_0}\right)(x - x_0) - \frac{x_0^2}{y_0^2}e^{x_0/y_0}(y - y_0) - (z - z_0) = 0,$$

代入 $x = y = z = 0$ 得到等式左边 $= -x_0e^{x_0/y_0} - \frac{x_0^2}{y_0}e^{x_0/y_0} + \frac{x_0^2}{y_0}e^{x_0/y_0} + z_0 = -x_0e^{x_0/y_0} + z_0 = 0 =$ 右边。 □

9.5.13 试给出正数 $\lambda > 0$, 使曲面 $xyz = \lambda$ 与 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 在某一点相切（有共同的切平面）。

解. 设

$$F_1(x, y, z) = xyz - \lambda, F_2(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1,$$
$$\boldsymbol{J}F_1 = (yz, xz, xy) = \left(\frac{\lambda}{x}, \frac{\lambda}{y}, \frac{\lambda}{z}\right), \boldsymbol{J}F_2 = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right).$$

由相切可得 $\boldsymbol{J}F_1$ 与 $\boldsymbol{J}F_2$ 平行，得到

$$k\left(\frac{\lambda}{x}, \frac{\lambda}{y}, \frac{\lambda}{z}\right) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right) \Rightarrow \frac{k\lambda}{2} = \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2},$$

代入 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ 得到

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}$$

结合$\lambda > 0$得到

$$\lambda = xyz = \sqrt{\frac{a^2}{3} \cdot \frac{b^2}{3} \cdot \frac{c^2}{3}} = \frac{abc}{3\sqrt{3}}.$$

□

9.5.16 求曲面 $x^2 + y^2 + z^2 = x$ 的切平面，使其垂直于平面 $x - y - z = 2$ 和 $x - y - z/2 = 2$.

解. $F(x, y, z) = x^2 + y^2 + z^2 - x, \mathbf{J}F = (2x - 1, 2y, 2z)$. 由于切平面垂直于平面 $x - y - z = 2$ 和 $x - y - z/2 = 2$, 得

$$\begin{cases} (2x-1, 2y, 2z) \perp (1, -1, -1) \\ (2x-1, 2y, 2z) \perp (1, -1, -\frac{1}{2}) \end{cases} \Rightarrow \begin{cases} 2x - 1 - 2y - 2z = 0 \\ 2x - 1 - 2y - z = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{2x-1}{2} \\ z = 0 \end{cases} \Rightarrow \boldsymbol{n} = \frac{1}{\sqrt{2}}(1, 1, 0)$$

为切平面的单位法向量，再求切平面与原曲面的交点。代入 $y=\frac{2x-1}{2}$ 和 $z=0$,

$$\begin{cases} x^2+y^2-x=0 \\ 2y=2x-1 \end{cases} \Rightarrow \begin{cases} x=\frac{2+\sqrt{2}}{4} \\ y=\frac{\sqrt{2}}{4} \end{cases} \quad \text{or} \quad \begin{cases} x=\frac{2-\sqrt{2}}{4} \\ y=-\frac{\sqrt{2}}{4} \end{cases},$$

代入并化简得满足条件的切平面为

$$x+y-\frac{1+\sqrt{2}}{2}=0 \text{ 和 } x+y-\frac{1-\sqrt{2}}{2}=0.$$

□

9.5.22 求 E,F,G.

(1) 椭球面：$\boldsymbol{r}(u,v)=(a\sin u\cos v, b\sin u\sin v, c\cos u)$;

(2) 单叶双曲面：$\boldsymbol{r}(u,v)=(a\cosh u\cos v, b\cosh u\sin v, c\sinh u)$;

(3) 椭圆抛物线：$\big(u,v,\frac{1}{2}\big(\frac{u^2}{a^2}+\frac{v^2}{b^2}\big)\big)$.

解.

(1) $\boldsymbol{r}_u=(a\cos u\cos v, b\cos u\sin v, -c\sin u), \boldsymbol{r}_v=(-a\sin u\sin v, b\sin u\cos v, 0)$;

$$E=||\boldsymbol{r}_u||^2=a^2\cos^2 u\cos^2 v+b^2\cos^2 u\sin^2 v+c^2\sin^2 u;$$
$$F=\langle \boldsymbol{r}_u,\boldsymbol{r}_v\rangle=(b^2-a^2)\cos u\sin u\cos v\sin v;$$
$$G=||\boldsymbol{r}_v||^2=a^2\sin^2 u\sin^2 v+b^2\sin^2 u\cos^2 v.$$

(2) $\boldsymbol{r}_u=(a\sinh u\cos v, b\sinh u\sin v, c\cosh u), \boldsymbol{r}_v=(-a\cosh u\sin v, b\cosh u\cos v, 0)$;

$$E=||\boldsymbol{r}_u||^2=a^2\sinh^2 u\cos^2 v+b^2\sinh^2 u\sin^2 v+c^2\cosh^2 u;$$
$$F=\langle \boldsymbol{r}_u,\boldsymbol{r}_v\rangle=(b^2-a^2)\sinh u\cosh u\sin v\cos v;$$
$$G=||\boldsymbol{r}_v||^2=a^2\cosh^2 u\sin^2 v+b^2\cosh^2 u\cos^2 v.$$

(3) $\boldsymbol{r}_u=(1,0,\dfrac{u}{a^2}), \boldsymbol{r}_v=(0,1,\dfrac{v}{b^2})$;

$$E=||\boldsymbol{r}_u||^2=1+\frac{u^2}{a^4};$$
$$F=\langle \boldsymbol{r}_u,\boldsymbol{r}_v\rangle=\frac{uv}{a^2b^2};$$
$$G=||\boldsymbol{r}_v||^2=1+\frac{v^2}{b^4}.$$

□

9.5.23 $I:=E\mathrm{d}u^2+2F\mathrm{d}u\mathrm{d}v+G\mathrm{d}v^2$, 证明：$I=\mathrm{d}\boldsymbol{r}^2$, 其中 $\boldsymbol{r}=\boldsymbol{r}(u,v)$.

解.

$$\begin{aligned} \mathrm{d}\boldsymbol{r}&=\boldsymbol{r}_u\mathrm{d}u+\boldsymbol{r}_v\mathrm{d}v \\ \mathrm{d}\boldsymbol{r}^2&=\langle \boldsymbol{r}_u\mathrm{d}u+\boldsymbol{r}_v\mathrm{d}v, \boldsymbol{r}_u\mathrm{d}u+\boldsymbol{r}_v\mathrm{d}v\rangle \\ &=||\boldsymbol{r}_u||^2\mathrm{d}u^2+2\boldsymbol{r}_u\boldsymbol{r}_v\mathrm{d}u\mathrm{d}v+||\boldsymbol{r}_v||^2\mathrm{d}v^2 \\ &=E\mathrm{d}u^2+2F\mathrm{d}u\mathrm{d}v+G\mathrm{d}v^2=I. \end{aligned}$$

□

9.5.24 已知曲面 $I = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2$, 求曲面上的曲线 $u = v$ 从 v_1 到 v_2 的弧长，其中 $v_2 > v_1$.

解. 由上题 $\mathrm{d}\boldsymbol{r}^2 = I = \mathrm{d}u^2 + (u^2 + a^2)\mathrm{d}v^2$, 在曲线 $u = v$ 上 $\mathrm{d}u = \mathrm{d}v$, 所以

$$\mathrm{d}\boldsymbol{r}^2 = (1 + a^2 + v^2)\mathrm{d}v^2 \Rightarrow |\mathrm{d}\boldsymbol{r}| = \sqrt{1 + a^2 + v^2}|\mathrm{d}v|,$$

在 v_1 到 v_2 上积分得弧长

$$s = \int_{v_1}^{v_2} |\mathrm{d}\boldsymbol{r}| = \int_{v_1}^{v_2} \sqrt{1 + a^2 + v^2}\mathrm{d}v = \frac{1}{2}v\sqrt{1 + a^2 + v^2} + \frac{1}{2}(1 + a^2)\log(v + \sqrt{1 + a^2 + v^2})\Big|_{v_1}^{v_2}.$$

□

9.6.1 计算 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
(2) $xy - \log y = 0$ 在 $(0, 1)$ 处；
(4) $x^y = y^x$.

解. (2) 隐函数定理：

$$\begin{aligned}
F(x, y) &= xy - \log y \\
\frac{\partial F}{\partial x} &= y, \frac{\partial F}{\partial y} = x - \frac{1}{y}, \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\partial F/\partial x}{\partial F/\partial y} = -\frac{y^2}{xy - 1}.
\end{aligned}$$

在 $(0, 1)$ 处有

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

或直接对等式两边求微分得

$$\begin{aligned}
x\mathrm{d}y + y\mathrm{d}x - \frac{1}{y}\mathrm{d}y &= 0 \\
(x - \frac{1}{y})\mathrm{d}y &= -y\mathrm{d}x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y^2}{xy - 1}
\end{aligned}$$

(4) 两边取对数得

$$\begin{aligned}
y\log x &= x\log y \\
\frac{y}{x}\mathrm{d}x + \log x\mathrm{d}y &= \frac{x}{y}\mathrm{d}y + \log y\mathrm{d}x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y/x - \log y}{x/y - \log x} = \frac{y^2 - x\log y}{x^2 - y\log x}
\end{aligned}$$

注：不取对数直接算出的结果

$$\frac{yx^{y-1} - y^x\log y}{xy^{x-1} - x^y\log x}$$

可通过等式 $x^y = y^x$ 化简后证明与上述结果一致。 □

9.6.2 计算 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.
(2) $\frac{x}{z} = \log\frac{z}{y}$;
(3) $(x + y + z)e^{-(x+y+z)}$;
(4) $z^2y - xz^3 - 1 = 0$, 在 $(1, 2, 1)$ 处;

解. (2)

$$\begin{aligned} \frac{\partial}{\partial x}: &\quad \frac{1}{z} - \frac{x}{z^2}\frac{\partial z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x} &&\Rightarrow \frac{\partial z}{\partial x} = \frac{z}{z+x}; \\ \frac{\partial}{\partial y}: &\quad -\frac{x}{z^2}\frac{\partial z}{\partial y} = \frac{1}{z}\frac{\partial z}{\partial y} - \frac{1}{y} &&\Rightarrow \frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}. \end{aligned}$$

(3)

$$\begin{aligned} \frac{\partial}{\partial x}: &\quad 1 + \frac{\partial z}{\partial x} = -e^{-(x+y+z)}\big(1 + \frac{\partial z}{\partial x}\big) &&\Rightarrow \frac{\partial z}{\partial x} = -1; \\ \frac{\partial}{\partial y}: &\quad 1 + \frac{\partial z}{\partial y} = -e^{-(x+y+z)}\big(1 + \frac{\partial z}{\partial y}\big) &&\Rightarrow \frac{\partial z}{\partial y} = -1. \end{aligned}$$

(4)

$$\begin{aligned} \frac{\partial}{\partial x}: &\quad 2yz\frac{\partial z}{\partial x} - z^3 - 3xz^2\frac{\partial z}{\partial x} = 0 &&\Rightarrow \frac{\partial z}{\partial x} = \frac{z^2}{2y-3xz}; \\ \frac{\partial}{\partial y}: &\quad z^2 + 2yz\frac{\partial z}{\partial y} - 3xz^2\frac{\partial z}{\partial y} = 0 &&\Rightarrow \frac{\partial z}{\partial y} = \frac{z}{2xz-2y}. \end{aligned}$$

代入点 $(1,2,1)$得

$$\frac{\partial z}{\partial x} = 1, \frac{\partial z}{\partial y} = -1.$$

□

9.6.3 设 $F(x,y,z) = 0$, 求证:

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

解. 将 x 看作关于 y,z 的函数 $x(y,z)$, 对 $F(x,y,z) = 0$ 两边求关于 y 的偏导得

$$F_x\frac{\partial x}{\partial y} + F_y = 0$$

解得

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x};$$

同理有

$$\begin{aligned} \frac{\partial y}{\partial z} &= -\frac{F_z}{F_y}; \\ \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z}. \end{aligned}$$

代入即得

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

□

9.6.4 设 $F(x-y, y-z, z-x) = 0$, 计算 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解. 将 z 看作关于 x,y 的函数 $z(x,y)$, 对等式两边分别求关于 x,y 的偏导,

$$\left\{ \begin{aligned} &F_1 - F_2\frac{\partial z}{\partial x} + F_3\big(\frac{\partial z}{\partial x} - 1\big) = 0 \\ &-F_1 + F_2\big(1 - \frac{\partial z}{\partial y}\big) + F_3\frac{\partial z}{\partial y} = 0 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} \frac{\partial z}{\partial x} &= \frac{F_3 - F_1}{F_3 - F_2} \\ \frac{\partial z}{\partial y} &= \frac{F_1 - F_2}{F_3 - F_2} \end{aligned} \right.$$

注：以上所有关于 F 的求导均在 $(x-y, y-z, z-x)$ 处，F_i 表示 F 对第 i 个分量求导。 □

9.6.5 设 $F(x+y+z,x^2+y^2+z^2)=0$, 计算 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解. 将 z 看作关于 x,y 的函数 $z(x,y)$, 对等式两边分别求关于 x,y 的偏导,

$$\begin{cases} F_1\big(1+\dfrac{\partial z}{\partial x}\big)+F_2\big(2x+2z\dfrac{\partial z}{\partial x}\big)=0 \\ F_1\big(1+\dfrac{\partial z}{\partial y}\big)+F_2\big(2y+2z\dfrac{\partial z}{\partial u}\big)=0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x}=-\dfrac{F_1+2xF_2}{F_1+2zF_2} \\ \dfrac{\partial z}{\partial y}=-\dfrac{F_1+2yF_2}{F_1+2zF_2} \end{cases}$$

注：以上所有关于 F 的求导均在 $(x+y+z,x^2+y^2+z^2)$ 处，F_i 表示 F 对第 i 个分量求导。 □

9.7.1 对方程

$$\begin{cases} x^2+y^2+z^2=1, \\ x+y+z=0, \end{cases}$$

计算 $\frac{dy}{dx}$ 和 $\frac{dz}{dx}$, 并作出解释.

解. 记 $F_1(x,y,z)=x^2+y^2+z^2-1$,$F_2(x,y,z)=x+y+z$. 由隐函数定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 y,z 关于 x 的函数 $(y,z)=\boldsymbol{f}(x)$. 且有

$$\begin{aligned} \mathbf{J}\boldsymbol{f}(x)=\begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{pmatrix} \\ &= -\begin{pmatrix} 2y & 2z \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2x \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{x-z}{y-z} \\ -\frac{x-y}{z-y} \end{pmatrix}. \end{aligned}$$

解释：这个曲线为一个圆。 □

9.7.3 对方程

$$\begin{cases} x=t+\frac{1}{t}, \\ y=t^2+\frac{1}{t^2}, \\ z=t^3+\frac{1}{t^3}, \end{cases}$$

计算 $\frac{dy}{dx}$ 和 $\frac{dz}{dx}$.

解. 带入 x,y,z 的方程有 $y=x^2-2$ 和 $z=x^3-3x$ 故求得

$$\frac{dy}{dx}=2x,$$
$$\frac{dz}{dx}=3x^2-3.$$

□

9.7.4 对下列方程, 计算 Jacobi 矩阵

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

(1) $xu-yv=0,yu+xv=1$;

(2) $x+y=u+v,\frac{x}{y}=\frac{\sin u}{\sin v}$.

解. (1) 记 $F_1(x,y,u,v)=xu-yv$,$F_2(x,y,u,v)=yu+xv-1$. 由隐映射定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 x,y 关于 u,v 的函数 $(x,y)=(x(u,v),y(u,v))$. 且有

$$\begin{aligned}
\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \\
&= -\begin{pmatrix} u & -v \\ v & u \end{pmatrix}^{-1} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \\
&= -\frac{1}{u^2+v^2}\begin{pmatrix} xu+yv & -yu+xv \\ yu-xv & xu+yv \end{pmatrix}.
\end{aligned}$$

(2) 记 $F_1(x,y,u,v)=x+y-u-v$,$F_2(x,y,u,v)=x\sin v-y\sin u$. 由隐映射定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 x,y 关于 u,v 的函数 $(x,y)=(x(u,v),y(u,v))$. 且有

$$\begin{aligned}
\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \\
&= -\begin{pmatrix} 1 & 1 \\ \sin v & -\sin u \end{pmatrix}^{-1} \begin{pmatrix} -1 & -1 \\ -y\cos u & x\cos v \end{pmatrix} \\
&= \frac{1}{\sin u+\sin v}\begin{pmatrix} y\cos u+\sin u & -x\cos v+\sin u \\ -y\cos u+\sin v & x\cos v+\sin v \end{pmatrix}.
\end{aligned}$$

□

9.7.6 设 $u=f(x,y,z,t), g(y,z,t)=0, h(z,t)=0$, 计算 $\frac{\partial u}{\partial x}$ 和 $\frac{\partial u}{\partial y}$.

解. 记 $F_1(y,z,t)=g(y,z,t)$,$F_2(z,t)=h(z,t)$. 由隐映射定理得 $\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}=0$ 确定了 z,t 关于 y 的函数 $(z,t)=\boldsymbol{f}(y)$. 且有

$$\begin{aligned}
\begin{pmatrix} \frac{\partial z}{\partial y} \\ \frac{\partial t}{\partial y} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial t} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial y} \end{pmatrix} \\
&= -\begin{pmatrix} g_2' & g_3' \\ h_1' & h_2' \end{pmatrix}^{-1} \begin{pmatrix} g_1' \\ 0 \end{pmatrix} \\
&= -\frac{1}{g_2'h_2'-g_3'h_1'}\begin{pmatrix} g_1'h_2' \\ -g_1'h_1' \end{pmatrix}.
\end{aligned}$$

由链式法则得

$$\frac{\partial u}{\partial x}=f_1'$$

$$\begin{aligned}
\frac{\partial u}{\partial y} &= f_2'+f_3'\frac{\partial z}{\partial y}+f_4'\frac{\partial t}{\partial y} \\
&= f_2'-\frac{g_1'(f_4'h_1'-f_3'h_2')}{g_2'h_2'-g_3'h_1'}
\end{aligned}$$

□

9.8.1 设 $D \subset \mathbb{R}^n$, 映射 $\boldsymbol{f} : D \to \mathbb{R}^n$. 如果 $\boldsymbol{f}$ 把开集映为开集, 则称 $\mathbf{J}$ 为一个开映射. 问下列映射是不是开映射:

(1) $\boldsymbol{f}(x,y) = \left(x^2, \frac{y}{x}\right)$;

(2) $\boldsymbol{f}(x,y) = (e^x \cos y, e^x \sin y)$;

(3) $\boldsymbol{f}(x,y) = x + y, 2xy^2$.

解. (1)

$$\mathbf{J}\boldsymbol{f}(x,y) = \begin{pmatrix} 2x & 0 \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix},$$

则计算得 $det\mathbf{J}\boldsymbol{f}(x,y) = 2 \neq 0$, 于是 f 为开映射.

(2)

$$\mathbf{J}\boldsymbol{f}(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix},$$

则计算得 $det\mathbf{J}\boldsymbol{f}(x,y) = e^{2x} \neq 0$, 于是 f 为开映射.

(3)

$$\mathbf{J}\boldsymbol{f}(x,y) = \begin{pmatrix} 1 & 1 \\ 2y^2 & 4xy \end{pmatrix},$$

则计算得 $det\mathbf{J}\boldsymbol{f}(x,y) = 2y(2x-y)$, $y \neq 0$ $x \neq 2x$ 时, 有 $det\mathbf{J}\boldsymbol{f}(x,y) \neq 0$, 于是 f 为开映射. □

9.8.2 对第题中的三个映射 $\boldsymbol{f}$, 计算 $\mathbf{J}\boldsymbol{f}^{-1}$.

解. (1)

$$\mathbf{J}\boldsymbol{f}(x,y)^{-1} = \begin{pmatrix} \frac{1}{2x} & 0 \\ \frac{y}{2x^2} & x \end{pmatrix},$$

(2)

$$\mathbf{J}\boldsymbol{f}(x,y)^{-1} = \begin{pmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{pmatrix},$$

(3)

$$\mathbf{J}\boldsymbol{f}(x,y)^{-1} = \frac{1}{2y(2x-y)} \begin{pmatrix} 4xy & -1 \\ -2y^2 & 1 \end{pmatrix},$$

□