Week12

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11.1.1 $\int_\Gamma (x^2+y^2)^n \mathrm{d}s, \Gamma: x=a\cos t, y=a\sin t (0\le t\le 2\pi)$.

解. $\mathrm{d}x=-a\sin t\mathrm{d}t, \mathrm{d}y=a\cos t\mathrm{d}t$,故

$$\int_\Gamma (x^2+y^2)^n \mathrm{d}s, \Gamma: x=a\cos t, y=a\sin t (0\le t\le 2\pi)=\int_0^{2\pi} a^{2n}|a|\mathrm{d}t=2\pi|a|^{2n+1}$$

□

11.1.2 $\int_\Gamma (x+y)\mathrm{d}s, \Gamma$:顶点为$(0,0),(1,0),(0,1)$的三角形的边界。

解.

$$\int_\Gamma (x+y)\mathrm{d}s=\int_0^1 x\mathrm{d}x+\int_0^1 \sqrt{2}\mathrm{d}x+\int_0^1 y\mathrm{d}y=1+\sqrt{2}$$

□

11.1.3 $\int_\Gamma z\mathrm{d}s, \Gamma$:圆锥螺线：$x=t\cos t, y=t\sin t, z=t(0\le t\le 2\pi)$.

解.

$$\mathrm{d}x=(\cos t-t\sin t)\mathrm{d}t, \mathrm{d}y=(\sin t+t\cos t)\mathrm{d}t, \mathrm{d}z=\mathrm{d}t.$$
$$\mathrm{d}s=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2}=\sqrt{2+t^2}\mathrm{d}t.$$
$$\int_\Gamma z\mathrm{d}s=\int_0^{2\pi} t\sqrt{2+t^2}\mathrm{d}t=\frac{1}{3}\big((2+4\pi^2)^{\frac{3}{2}}-2\sqrt{2}\big).$$

□

11.1.4 $\int_\Gamma x^2\mathrm{d}s, \Gamma$:圆周$x^2+y^2+z^2=a^2, x+y+z=0$.

解.

$$\int_\Gamma x^2\mathrm{d}s=\frac{1}{3}\int_\Gamma (x^2+y^2+z^2)\mathrm{d}s=\frac{a^2}{3}\int_\Gamma \mathrm{d}s=\frac{2}{3}\pi|a|^3.$$

□

11.1.5 $\int_\Gamma y^2\mathrm{d}s, \Gamma$:旋轮线的一拱，$x=a(t-\sin t), y=a(1-\cos t)(0\le t\le 2\pi)$.

解.

$$\mathrm{d}x=a(1-\cos t)\mathrm{d}t, \mathrm{d}y=a\sin t\mathrm{d}t.$$
$$\int_\Gamma y^2\mathrm{d}s=\int_\Gamma a^2(1-\cos t)^2 2|a\sin\frac{t}{2}|\mathrm{d}t=8|a|^3\int_0^{2\pi}\sin^5\frac{t}{2}\mathrm{d}t=\frac{256}{15}|a|^3.$$

□

11.2.1 计算第二型曲线积分。

(1) $\displaystyle\int_\Gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2}$,$\Gamma$表示逆时针方向的圆周$x^2+y^2=a^2$;

解. $x = a\cos t, y = a\sin t$,故$\mathrm{d}x = -a\sin t\mathrm{d}t, \mathrm{d}y = a\cos t dt$,方向为$t$增加的方向,

$$\frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} = \int_0^{2\pi} \frac{a^2\cos^2 t\mathrm{d}t + a^2\sin^2 t\mathrm{d}t}{a^2} = 2\pi.$$

□

(3) $\displaystyle\int_\Gamma (x^2-2xy)\mathrm{d}x + (y^2-2xy)\mathrm{d}y, \Gamma : x = y^2(-1 \le y \le 1)$,沿$y$增加的方向;

解.

$$\int_\Gamma (x^2-2xy)\mathrm{d}x + (y^2-2xy)\mathrm{d}y = \int_{-1}^1 (y^4-2y^3)2y\mathrm{d}y + (y^2-2y^3)\mathrm{d}y = -\frac{14}{15}.$$

□

(5) $\displaystyle\int_\Gamma (x^2+y^2)\mathrm{d}y$,$\Gamma$是直线$x=1, x=3$和$y=1, y=4$构成的矩形,沿逆时针方向。

解.

$$\int_\Gamma (x^2+y^2)\mathrm{d}y = \int_1^4 (9+y^2)\mathrm{d}y + \int_4^1 (1+y^2)\mathrm{d}y = 24.$$

□

11.2.2 设常数a,b,c满足$ac-b^2 > 0$.计算$\displaystyle\int_\Gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{ax^2+2bxy+cy^2}$,其中$\Gamma$为逆时针方向的单位圆周。

解. $x = \cos t, y = \sin t$,故$\mathrm{d}x = -\sin t\mathrm{d}t, \mathrm{d}y = \cos t\mathrm{d}t$.

$$\begin{aligned}
\int_\Gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{ax^2+2bxy+cy^2} &= \int_0^{2\pi} \frac{\mathrm{d}t}{a\cos^2 t + 2b\cos t\sin t + c\sin^2 t} \\
&= 2\int_{-\pi/2}^{\pi/2} \frac{1}{a+2b\tan t + c\tan^2 t}\cdot\frac{\mathrm{d}t}{\cos^2 t} \\
&= 2\int_{-\infty}^{+\infty} \frac{1}{a+2bt+ct^2}\mathrm{d}t \\
&= \frac{2\pi\mathrm{sgn}(c)}{\sqrt{ac-b^2}}
\end{aligned}$$

□

注. 答案是$sgn(a)$也可.

11.2.3 计算第二型曲线积分,曲线的正向是参数增加的方向;

(1) $\displaystyle\int_\Gamma xz^2\mathrm{d}x + yx^2\mathrm{d}y + zy^2\mathrm{d}z, \Gamma : x = t, y = t^2, z = t^3 (0 \le t \le 1)$;

解.

$$\int_\Gamma xz^2\mathrm{d}x + yx^2\mathrm{d}y + zy^2\mathrm{d}z = \int_0^1 t^7 + 2t^5 + 3t^9\mathrm{d}t = \frac{91}{120}.$$

□

(2) $\int_\Gamma (y+z)\mathrm{d}x+(z+x)\mathrm{d}y+(x+y)\mathrm{d}z, \Gamma: x=a\sin^2 t, y=2a\sin t\cos t, z=a\cos^2 t(0\le t\le\pi)$.

解. $\mathrm{d}x=2a\sin t\cos t\mathrm{d}t, \mathrm{d}y=2a(\cos^2 t-\sin^2 t), \mathrm{d}z=-2a\sin t\cos t$.

$$\begin{aligned}
&\int_\Gamma (y+z)\mathrm{d}x+(z+x)\mathrm{d}y+(x+y)\mathrm{d}z\\
=&a^2\int_0^\pi (2\sin t\cos t+\cos^2 t)\cdot 2\sin t\cos t+2(\cos^2 t-\sin^2 t)-2\sin t\cos t(\sin^2 t+2\sin t\cos t)\mathrm{d}t\\
=&a^2\int_0^\pi \cos 2t(2+\sin 2t)\mathrm{d}t\\
=&0
\end{aligned}$$

□

11.2.4 $\int_\Gamma (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z, \Gamma$为球面片$x^2+y^2+z^2=1, x\ge 0, y\ge 0, z\ge 0$的边界，方向是$(1,0,0)\to(0,1,0)\to(0,0,1)\to(1,0,0)$.

解. 设题中定义的曲线$(1,0,0)\to(0,1,0)$段为Γ_1,$(0,1,0)\to(0,0,1)$段为Γ_2,$(0,0,1)\to(1,0,0)$段为Γ_3.

$$\int_{\Gamma_1}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z=\int_{\Gamma_1}y^2\mathrm{d}x-x^2\mathrm{d}y=\int_0^{\pi/2}-\sin^3 t-\cos^3 t\mathrm{d}t=-\frac{4}{3}$$

同理，有

$$\int_{\Gamma_2}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z=\int_{\Gamma_3}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z=\int_{\Gamma_1}(y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-$$

因此

$$\int_\Gamma (y^2-z^2)\mathrm{d}x+(z^2-x^2)\mathrm{d}y+(x^2-y^2)\mathrm{d}z=-4$$

□

11.2.5 证明：$\Big|\int_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{p}\Big|\le\int_\Gamma ||\boldsymbol{F}||\mathrm{d}s$.

解.

$$\Big|\int_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{p}\Big|=\Big|\int_\Gamma\sum_{i=1}^n F_i\cdot\mathrm{d}p_i\Big|\le\int_\Gamma\sum_{i=1}^n|F_i|\cdot|\mathrm{d}p_i|\le\int_\Gamma\Big(\sum_{i=1}^n|F_i|^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^n|\mathrm{d}p_i|^2\Big)^{\frac{1}{2}}=\int_\Gamma||\boldsymbol{F}||\mathrm{d}s$$

□

11.3.1 利用Green公式计算

(1) $\int_\Gamma xy^2\mathrm{d}y-x^2y\mathrm{d}x, \Gamma$为圆周$x^2+y^2=a^2$,按逆时针方向；

(2) $\int_\Gamma (x+y)\mathrm{d}x-(x-y)\mathrm{d}y, \Gamma$为椭圆$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,按逆时针方向；

(3) $\int_\Gamma e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y, \Gamma$为上半圆周$x^2+y^2=ax$沿$x$增加的方向。

解. (1)$P(x,y)=-x^2y,Q(x,y)=xy^2$,

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y^2+x^2$$

由Green公式

$$\int_\Gamma xy^2\mathrm{d}y-x^2y\mathrm{d}x=\iint_{B_a(0)}x^2+y^2\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\int_0^a r^3\mathrm{d}r\mathrm{d}\theta=\frac{\pi a^4}{2}.$$

(2)$P(x,y)=x+y,Q(x,y)=y-x$,

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-2$$

由Green公式

$$\int_\Gamma(x+y)\mathrm{d}x-(x-y)\mathrm{d}y=\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}-2\mathrm{d}x\mathrm{d}y=-2\pi ab.$$

(3)设$\Gamma_1=-\Gamma,\Gamma_2=\{(x,0)|x:0\to a\}$,则$P(x,y)=e^x\sin y,Q(x,y)=e^x\cos y$,

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$$

由Green公式

$$\int_{\Gamma_1+\Gamma_2}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=\iint_{x^2+y^2\le ax,y\ge 0}0\mathrm{d}x\mathrm{d}y=0,$$

又在Γ_2上$y=0,\mathrm{d}y=0$故

$$\int_{\Gamma_2}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=0.$$

因此

$$\int_\Gamma e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=-\int_{\Gamma_1}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=-\int_{\Gamma_1+\Gamma_2}e^x\cos y\mathrm{d}x+e^x\cos y\mathrm{d}y=0.$$

□

11.3.2 用Green公式计算面积。

(1) 星形线$x=a\cos^3 t,y=a\sin^3 t(0\le t\le 2\pi)$;

(2) 双纽线$(x^2+y^2)^2=a^2(x^2-y^2)$;

(3) Descartes叶形线$x^3+y^3=3axy(a>0)$.

解.

$$\sigma(D)=\int_{\partial D}x\mathrm{d}y=-\int_{\partial D}y\mathrm{d}x=\frac{1}{2}\int_{\partial D}x\mathrm{d}y-y\mathrm{d}x.$$

(1)$\mathrm{d}x=-3a\cos^2 t\sin t\mathrm{d}t,\mathrm{d}y=3a\cos t\sin^2 t\mathrm{d}t$,

$$\begin{aligned}\sigma(D)&=\frac{1}{2}\int_{\partial D}x\mathrm{d}y-y\mathrm{d}x\\&=\frac{1}{2}\int_{\partial D}(3a^2\cos^4 t\sin^2 t+3a^2\cos^2 t\sin^4 t)\mathrm{d}t\\&=\frac{3a^2}{2}\int_{\partial D}\cos^2 t\sin^2 t\mathrm{d}t\\&=\frac{3\pi a^2}{8}\end{aligned}$$

(2)令$y=x\tan\theta$,代入方程得$\big(x^2(1+\tan^2\theta)\big)^2=a^2(1-\tan^2\theta)x^2$,

$$x^2=\frac{a^2(1-\tan^2\theta)}{(1+\tan^2\theta)^2}=a^2\cos^2\theta\cos(2\theta),y^2=a^2\sin^2\theta\cos(2\theta)$$

θ范围：由计算知$\cos(2\theta)\geq 0$,故$\theta\in[0,\frac{\pi}{4}]\cup[\frac{3\pi}{4},\frac{5\pi}{4}]\cup[\frac{7}{4\pi},2\pi]$. 由对称性，只考虑第一象限即可。

$$\begin{aligned}y\mathrm{d}x=x\tan\theta\mathrm{d}x=&a^2\tan\theta\big(-2\cos\theta\sin\theta\cos(2\theta)-2\cos^2\theta\sin(2\theta)\big)\mathrm{d}\theta\\=&-2a^2\big(\sin^2\theta\cos(2\theta)+\sin\theta\cos\theta\sin(2\theta)\big)\mathrm{d}\theta\\x\mathrm{d}y=\frac{y}{\tan\theta}\mathrm{d}y=&\frac{a^2}{\tan\theta}\big(2\sin\theta\cos\theta\cos(2\theta)-2\sin^2\theta\sin(2\theta)\big)\mathrm{d}\theta\\=&2a^2\big(\cos^2\theta\cos(2\theta)-\sin\theta\cos\theta\sin(2\theta)\big)\mathrm{d}\theta\end{aligned}$$

代入得

$$\frac{1}{4}\sigma(D)=\frac{1}{8}\int_{\partial D}x\mathrm{d}y-y\mathrm{d}x=\frac{1}{8}\int_0^{\pi/4}2a^2\cos(2\theta)\mathrm{d}\theta=\frac{a^2}{4},$$

故

$$\sigma(D)=a^2.$$

(3) 令$y=xt$,代入方程得$x^3(1+t^3)=3ax^2t$,得到参数方程

$$x=\frac{3at}{1+t^3},y=\frac{3at^2}{1+t^3},$$

用参数$u=\frac{1}{t}$代换可得上述x,y交换的表达式，容易知道上述用t作为参数的表达式在$t\in[0,1]$时是从$(0,0)$到$\Big(\frac{3a}{2},\frac{3a}{2}\Big)$的曲线$\Gamma_1$，而在参数$u\in[0,1]$时是与$\Gamma_1$关于$y=x$对称的曲线$\Gamma_2$，因此$\Gamma_1$和$-\Gamma_2$构成了封闭曲线，其参数范围$t\in[0,+\infty)$.

$$\mathrm{d}x=\frac{3a(1-2t^3)}{1+t^3)^2}\mathrm{d}t$$

由Green公式，换元$u=t^3$,

$$\begin{aligned}\sigma(D)=-\int_{\partial}y\mathrm{d}x=9a^2\int_0^{+\infty}\frac{t^2(2t^3-1)}{(1+t^3)^3}\mathrm{d}t&=3a^2\int_0^{+\infty}\frac{2u-1}{(1+u)^3}\mathrm{d}u\\&=3a^2\int_0^{+\infty}\frac{2}{(1+u)^2}-\frac{3}{(1+u)^3}\mathrm{d}u\\&=\frac{3a^2}{2}.\end{aligned}$$

□

11.3.3 封闭曲线$\Gamma:x=\varphi(t),y=\psi(t)(\alpha\leq t\leq\beta)$,参数增加的方向是正方向，求证$\Gamma$围城的面积

$$A=\frac{1}{2}\int_\alpha^\beta\begin{vmatrix}\varphi(t)&\psi(t)\\\varphi'(t)&\psi'(t)\end{vmatrix}\mathrm{d}t.$$

解.

$$\mathrm{d}x=\varphi'(t)\mathrm{d}t,\mathrm{d}y=\psi'(t)\mathrm{d}t$$

故

$$A = \frac{1}{2}\int_{\partial D} x\mathrm{d}y - y\mathrm{d}x = \frac{1}{2}\int_{\alpha}^{\beta} \begin{vmatrix} \varphi(t) & \psi(t) \\ \varphi'(t) & \psi'(t) \end{vmatrix} \mathrm{d}t.$$

$\square$